The Perturbation Kinetic Energy Budget

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Motivation

Atmospheric motions can be thought of as a cycle of energy that starts from diabatic heating (i.e. solar radiation) and effectively lifts the atmospheric center of mass. This creates and maintains reservoirs of potential energy that can be converted into kinetic energy (KE), and eventually transferred into the internal energy of individual molecules and radiated away. Kinetic energy is a useful quantity to study that is different from momentum because it is a scalar quantity, rather than a vector. Another important difference is that it increases with the square of the velocity. Kinetic energy is especially useful for studying the growth and propagation of wave disturbancess in the atmosphere.

$$K = \frac{\vec{V} \cdot \vec{V}}{2}$$

We are often interested in disturbances of a certain scale and how those disturbances interact with features organized on larger scales. Thus it is useful to split kinetic energy up into different scales, either in time or space, or both. In the atmospheric dynamics literature this is usually done in the spatial sense by separating the zonal mean field from the "eddies". When analyzing more localized features, such as easterly waves, it is arguably more useful to separate the temporally slow and fast components of the kinetic energy field. We will refer to these components as the *background* kinetic energy (BKE) and *perturbation* kinetic energy (PKE).

$$K = K_B + K_P \tag{1}$$

We can write this decomposition in terms of the momentum vector.

$$K = \frac{1}{2}\left(\overline{\vec{V}} + \vec{V}'\right) \cdot \left(\overline{\vec{V}} + \vec{V}'\right) \tag{2}$$

$$= \frac{1}{2}\left(\overline{\vec{V}} \cdot \overline{\vec{V}} + 2\vec{V}' \cdot \overline{\vec{V}} + \vec{V}' \cdot \vec{V}'\right) \tag{3}$$

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This decomposition is relatively simple when the overbar in the above expression is a simple time average, which results in a time-invariant quantity. However, when the overbar is a filter this approach is complicated by non-linear terms that cannot be simplified with traditional rules of Reynold's averaging. This also means that these is no direct relationship between the terms in (1) and (2).

Even in the case of an impossibly *perfect* filter, we run into odd identities. For example, for some generic variable $x(t)$,

$$\overline{(x'\overline{x})} \neq 0,$$

due to the non-linear nature of the product. To understand this in the simplest case, consider $\overline{x}$ and x' to each represent a single sine wave with frequencies above and below some cutoff frequency. The product of $\overline{x}$ and x' results in two new frequencies that are the sum and difference of the input frequencies. The sum of the frequencies is certainly above the cutoff, but there's no way to be sure whether the difference is above or below the cutoff. This non-linear interaction becomes even more complicated when we consider a spectrum of waves.

$$\sin\alpha \sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

In light of this caveat the budget for PKE cannot be exact. We will define the perturbation momentum budget as a first step to make the budget of PKE tractable. This inherently assumes that the budgets of low and high frequency momentum are separable. In practice this assumption does not cause substantial errors, as the closure of the total and perturbation momentum budgets have similar residuals (not shown).

PKE Budget Derivation

This derivation was adapted from several previous papers (Lau and Lau, 1992; Maloney and Dickinson, 2003; Alaka and Maloney, 2014). To derive a budget for PKE we will start with the approximate budgets of zonal and meridional momentum,

$$\partial_t u = -\vec{V}\cdot\nabla u + fv - \partial_x\phi + D_x, \tag{4}$$

$$\partial_t v = -\vec{V}\cdot\nabla v - fu - \partial_y\phi + D_y, \tag{5}$$

where $\phi = gz$ is the geopotential, f is the coriolis parameter, and D_x and D_y represent frictional dissipation, including unresolved processes such as convective momentum transport. In practice the dissipation is calculated as a residual will also includes errors in the estimation of other terms. This residual is often of a similar magnitude to the other terms, and so it should not be neglected.

Notice that the wind vector and gradient operator are 3-dimensional vectors.

$$\vec{V} = \hat{i}u + \hat{j}v + \hat{k}\omega$$
$$\nabla = \hat{i}\partial_x + \hat{j}\partial_y + \hat{k}\partial_p$$

We first need to obtain the perturbation momentum budgets from the total momentum budgets (4) and (5). To do this we can write each variable as the sum of a long term mean and perturbation.

$$\begin{aligned} \partial_t \left(\overline{u} + u'\right) = &- \left(\overline{\vec{V}} + \vec{V}'\right) \cdot \nabla \left(\overline{u} + u'\right) \\ &+ f\left(\overline{v} + v'\right) - \partial_x \left(\overline{\phi} + \phi'\right) + \left(\overline{D_x} + D_x'\right) \end{aligned} \tag{6}$$

$$\begin{aligned} \partial_t \left(\overline{v} + v'\right) = &- \left(\overline{\vec{V}} + \vec{V}'\right) \cdot \nabla \left(\overline{v} + v'\right) \\ &- f\left(\overline{u} + u'\right) - \partial_y \left(\overline{\phi} + \phi'\right) + \left(\overline{D_y} + D_y'\right) \end{aligned} \tag{7}$$

Applying a time mean to (6) and (7) gives a budget time mean momentum.

$$\partial_t \overline{u} = -\overline{\vec{V}} \cdot \nabla \overline{u} - \overline{\vec{V}' \cdot \nabla u'} + f\overline{v} - \partial_x \overline{\phi} + \overline{D_x} \tag{8}$$
$$\partial_t \overline{v} = -\overline{\vec{V}} \cdot \nabla \overline{v} - \overline{\vec{V}' \cdot \nabla v'} - f\overline{u} - \partial_y \overline{\phi} + \overline{D_y} \tag{9}$$

Notice that we could set the local time tendency terms to zero in (8) and (9) if the time mean is sufficiently long, but these terms have been left intact to maintain generality. Subtracting (8) and (9) from (6) and (7) gives the budgets of perturbation momentum.

$$\partial_t u' = -\overline{\vec{V}} \cdot \nabla u' - \vec{V}' \cdot \nabla u' - \vec{V}' \cdot \nabla \overline{u} + \overline{\vec{V}' \cdot \nabla u'} + fv' - \partial_x \phi' + D_x', \tag{10}$$
$$\partial_t v' = -\overline{\vec{V}} \cdot \nabla v' - \vec{V}' \cdot \nabla v' - \vec{V}' \cdot \nabla \overline{v} + \overline{\vec{V}' \cdot \nabla v'} - fu' - \partial_y \phi' + D_y', \tag{11}$$

Next we will multiply (10) by u' and (11) by v' to obtain budgets of the zonal and meridional perturbation kinetic energy.

$$\begin{aligned} \partial_t \frac{u'^2}{2} = &- \overline{\vec{V}} \cdot \nabla \frac{u'^2}{2} - \vec{V}' \cdot \nabla \frac{u'^2}{2} \\ &- u'\vec{V}' \cdot \nabla \overline{u} + u'\overline{\vec{V}' \cdot \nabla u'} + fu'v' - u'\partial_x \phi' + D_x \end{aligned} \tag{12}$$

$$\begin{aligned} \partial_t \frac{v'^2}{2} = &- \overline{\vec{V}} \cdot \nabla \frac{v'^2}{2} - \vec{V}' \cdot \nabla \frac{v'^2}{2} \\ &- v'\vec{V}' \cdot \nabla \overline{v} + v'\overline{\vec{V}' \cdot \nabla v'} - fu'v' - v'\partial_y \phi' + D_y \end{aligned} \tag{13}$$

We can now add (12) and (13) to cancel the coriolis toque and obtain a budget equation for PKE.

$$\begin{aligned}\partial_t\left(\frac{u'^2+v'^2}{2}\right) =& -\overline{\vec{V}}\cdot\nabla\left(\frac{u'^2+v'^2}{2}\right)-\vec{V}'\cdot\nabla\left(\frac{u'^2+v'^2}{2}\right)\\ &-\vec{V}_H'\cdot\left(\vec{V}'\cdot\nabla\right)\overline{\vec{V}_H}+\vec{V}_H\cdot\overline{\vec{V}'\cdot\nabla V'}-\vec{V}_H'\cdot\nabla_H\phi'+D.\end{aligned} \tag{14}$$

Notice that subscript H is used to indicate when only horizontal components of a vector are considered.

Some further modification is useful. The last term on the LHS of (14) can be expanded using the product rule.

$$\vec{V}_H'\cdot\nabla_H\phi' = -\phi'\nabla_H\cdot\vec{V}_H'+\nabla_H\cdot\left(\vec{V}_H'\phi'\right) \tag{15}$$

We can then substitute for the wind divergence on the RHS of (15) with the help of the continuity equation,

$$\frac{du}{dx}+\frac{dv}{dy}=\frac{d\omega}{dp},$$

to get,

$$\vec{V}_H'\cdot\nabla_H\phi' = -\phi'\partial_p\omega'+\nabla_H\cdot\left(\vec{V}_H'\phi'\right). \tag{16}$$

Another use of the product rule can expand the first term on the RHS of (16).

$$\vec{V}_H'\cdot\nabla_H\phi' = -\omega'\partial_p\phi'+\partial_p\left(\omega'\phi'\right)+\nabla_H\cdot\left(\vec{V}_H'\phi'\right) \tag{17}$$

One last manipulation is to invoke the approximation of hydrostatic balance,

$$\frac{\partial p}{\partial z}=-\rho g,$$

rewritten in a more relevant form along with the ideal gas law,

$$\frac{\partial\phi'}{\partial p}=-\frac{1}{\rho'}=-\frac{R_dT'}{p}. \tag{18}$$

Substituting (18) and (17) into (14) along with a definition of PKE,

$$K_P=\frac{u'^2+v'^2}{2},$$

gives us the final budget equation for PKE.

$$\begin{aligned}\boldsymbol{\partial_t K_P =} & \boldsymbol{-\overline{\vec{V}}\cdot\nabla K_P\ -\vec{V}'\cdot\nabla K_P}\\ & \boldsymbol{\left[-\vec{V}_H'\cdot\left(\vec{V}'\cdot\nabla\right)\overline{\vec{V}_H}+\vec{V}_H\cdot\overline{\vec{V}'\cdot\nabla V'}\right]\ -\frac{R_d\omega'T'}{p}\ -\nabla\cdot\left(\vec{V}'\phi'\right)\ +D}\end{aligned} \tag{19}$$

The terms of (19) are summarized in the table below. The first two terms on the RHS of (19) describe advection of PKE. The remaining terms describe how PKE is created and destroyed. PKE must ultimately originate from a generation of available potential energy (Lorenz, 1955). Generation of low-frequency background available potential energy (BAPE) will be converted to BKE, where it can be further converted to PKE through barotropic conversion. The fourth term of on the RHS of (19) describes conversion of perturbation available potential energy (PAPE) to PKE through baroclinic overturning circulations.

Budget Term	Shorthand	Description
$\partial_t K_P$		Local Eulerian PKE tendency
$-\overline{\vec{V}} \cdot \nabla K_P$	A_B	PKE advection by the background flow
$-\vec{V}' \cdot \nabla K_P$	A_P	PKE advection by the perturbation flow
$\left[-\vec{V}'_H \cdot \left(\vec{V}' \cdot \nabla \right) \overline{\vec{V}_H} + \vec{V}_H \cdot \overline{\vec{V}' \cdot \nabla V'} \right]$	B_T	Barotropic conversion of BKE to PKE
$-\frac{R_d \omega' T'}{p}$	B_C	Baroclinic conversion of PAPE to PKE
$-\nabla \cdot \left(\vec{V}' \phi' \right)$	ϕ_F	Geopotential flux convergence
D		Residual PKE source/sink (i.e. friction, CMT)

Table 1: Description of terms in the PKE budget equation (19).

References

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