The Perturbation Available Potential Energy Budget

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Motivation

Potential and internal energy are the only two direct sources of kinetic energy in the atmosphere. The sum of geopotential and internal energy is referred to as the *total potential energy* (TPE), which is equal to the total enthalpy of the system.

$$TPE = P + I = \int_0^{p_s} (gz + c_v T)\frac{dp}{g} = \int_0^{p_s} c_p T \frac{dp}{g}$$

In adiabatic, inviscid flow the total energy is converted between geopotential, internal, and kinetic energy, but is conserved overall. The dry static energy (DSE) is defined as the sum of enthalpy and geopotential energy, and is conserved for adiabatic parcel displacements.

$$s = c_p T + gz \tag{1}$$

It is easy to see that it is impossible for all potential energy to be converted into kinetic energy, since this would result in an atmosphere where all the mass resided in an infinitely small layer at the surface. Thus there is a limit to how much potential energy is available to be converted to kinetic, but how can we find this limit?

Lorenz (1955) showed that this available potential energy (APE) could be quantified by considering a state in which the mass of the atmosphere was adiabatically rearranged to minimize the total potential energy of the system. In such a state the atmosphere would be statically stable everywhere, and all isentropic surfaces would be parallel to the ground. By arranging the mass adiabatically we know that every parcel will maintain their potential temperature (and DSE), and so the estimate of APE really comes down to estimating the variance of mass on isentropic surfaces.

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PAPE Budget Derivation

The processes that convert energy to and from APE are useful for explaining the kinetic energy field. In the case of transient tropical wave disturbances we are specifically interested in the perturbation kinetic energy (PKE) and the perturbation available potential energy (PAPE).

The derivation of the PAPE budget begins with the DSE budget,

$$\partial_t s + \vec{V}_H \cdot \nabla_H s + \omega \partial_p s = Q_1, \tag{2}$$

where Q_1 is the residual of the budget, which accounts for diabatic sources and sinks. Q_1 is most notably dominated by condensation and evaporation associated with moist convection. Radiation also contributes to Q_1, most often as a net cooling.

Many published studies have approximated the DSE budget by ignoring the Eulerian and advective tendencies of the geopotential. The justification for this approach is not clear, but to be consistent with the literature we will continue with this simplified budget.

$$c_p \partial_t T + c_p \vec{V}_H \cdot \nabla_H T + \omega \partial_p s = Q_1, \tag{3}$$

The first step is to separate the terms into means and perturbations.

$$\partial_t(\overline{T} + T') + (\overline{\vec{V}_H} + \vec{V}_H') \cdot \nabla_H(\overline{T} + T') + (\overline{\omega} + \omega')\partial_p(\overline{T} + T') = (\overline{Q_1} + Q_1') \tag{4}$$

Taking the time mean of (4) gives us an expression for the mean DSE budget.

$$\partial_t \overline{T} + \overline{\vec{V}_H} \cdot \nabla_H \overline{T} + \overline{\vec{V}_H' \cdot \nabla_H T'} + \overline{\omega} \partial_p \overline{s} + \overline{\omega' \partial_p s'} = \overline{Q_1} \tag{5}$$

We can then subtract (5) from (4) to get the budget of perturbation DSE.

$$\begin{aligned} c_p \partial_t T' + c_p \overline{\vec{V}_H} \cdot \nabla_H T' \quad & + c_p \vec{V}_H' \cdot \nabla_H \overline{T} \quad + c_p \left(\vec{V}_H' \cdot \nabla_H T' \right)' \\ + \overline{\omega} \partial_p s' \quad & + \omega' \partial_p \overline{s} \quad + \left(\omega' \partial_p s' \right)' \quad = Q_1' \end{aligned} \tag{6}$$

A useful simplification is to ignore terms that do not contribute significantly, such as the perturbation of a product of perturbations. Terms involving the horizontal gradient of a temperature perturbation can be ignored on the grounds that they are generally weak in the Tropics (Sobel et al., 2001; Romps, 2012). The vertical advection of the perturbation DSE by the mean vertical motion can also be ignored because the mean vertical motion is small.

$$
\begin{aligned}
c_p \partial_t T' + \cancel{c_p \overline{\vec{V}_H} \cdot \nabla_H T'} \quad &+ c_p \vec{V}_H' \cdot \nabla_H \overline{T} \quad + \cancel{c_p \left(\vec{V}_H' \cdot \nabla_H T' \right)'} \\
+ \cancel{\overline{\omega} \partial_p s'} \quad &+ \omega' \partial_p \overline{s} \quad + \cancel{\left(\omega' \partial_p s'\right)'} \qquad = Q_1'
\end{aligned} \tag{7}
$$

This leaves a much simpler equation to work with.

$$
c_p \partial_t T' = Q_1' - c_p \vec{V}_H' \cdot \nabla_H \overline{T} - \omega' \partial_p \overline{s} \tag{8}
$$

The vertical gradient of $\overline{s}$ is one way to measure the static stability of the atmosphere, but with some manipulation we can rewrite this in somewhat more familiar terms. Using the hydrostatic equation,

$$
dp = -\rho g dz,
$$

the vertical gradient of $\overline{s}$ can be rewritten as,

$$
\begin{aligned}
\partial_p \overline{s} &= c_p \frac{\partial \overline{T}}{\partial p} + g \frac{\partial \overline{z}}{\partial p} \\
&= \frac{c_p}{-\overline{\rho} g} \frac{\partial \overline{T}}{\partial z} - \frac{1}{\overline{\rho}} \\
&= \frac{c_p}{-\overline{\rho} g} \left(\frac{\partial \overline{T}}{\partial z} + \frac{g}{c_p} \right).
\end{aligned} \tag{9}
$$

We can then substitute Γ for the temperature lapse rate as well as the definition of the dry adiabatic lapse rate,

$$
\Gamma_d = \frac{g}{c_p},
$$

to obtain,

$$
\partial_p \overline{s} = -\frac{\Gamma_d - \Gamma}{\Gamma_d \overline{\rho}} = -\frac{1}{\gamma \overline{\rho}}, \tag{10}
$$

where γ is the inverted static stability,

$$
\gamma = \frac{\Gamma_d}{\Gamma_d - \Gamma} \tag{11}
$$

Substituting (10) into (8) gives,

$$
c_p \partial_t T' = Q_1' - c_p \vec{V}_H' \cdot \nabla_H \overline{T} + \frac{\omega'}{\gamma \overline{\rho}} \tag{12}
$$

The next step is to multiply (12) by $\frac{\gamma T'}{\overline{T}}$.

$$
\partial_t \frac{\gamma c_p T'^2}{2\overline{T}} = \frac{\gamma T' Q_1'}{\overline{T}} - \frac{\gamma c_p}{\overline{T}} \left(\vec{V}_H' T' \right) \cdot \nabla_H \overline{T} + \frac{\omega' T'}{\overline{\rho} \overline{T}} \tag{13}
$$

The ideal gas law,

$$p = \rho R T,$$

can be used to make the last term in (13) more convenient for data on pressure levels.

$$\partial_t \frac{\gamma c_p T'^2}{2\overline{T}} = \frac{\gamma T' Q_1'}{\overline{T}} - \frac{\gamma c_p}{\overline{T}} \left(\vec{V}_H' T' \right) \cdot \nabla_H \overline{T} + \frac{R}{\overline{p}} \omega' T' \tag{14}$$

Finally, we can define PAPE as

$$A_P = \gamma c_p \frac{T'^2}{2\overline{T}} \tag{15}$$

and apply this definition to (14) to obtain an expression for the budget of PAPE.

$$\partial_t A_P = \frac{\gamma}{\overline{T}} T' Q_1' - \frac{\gamma c_p}{\overline{T}} \left(\vec{V}_H' T' \right) \cdot \nabla \overline{T} + \frac{R}{\overline{p}} \omega' T' \tag{16}$$

The terms on the RHS of (16) describe the sources and sinks of PAPE and are summarized in the table below. The first term represents the generation of PAPE by diabatic processes, which is positive when diabatic heating is positively correlated with temperature anomalies. In other words, heating where it is warm, or cooling where it is cold, raises the center of mass of the atmosphere. The second term involves horizontal eddy fluxes across a background temperature gradient, and thus represents the baroclinic conversion of mean APE to PAPE. This term is positive when these eddies fluxes act to sharpen the background temperature gradient, and negative when they act to smooth the background temperature gradient. The third term is the conversion of PAPE to PKE by baroclinic overturning circulations. This conversion is generally negative because warm air is ascending or cold air is sinking, which lowers the center of mass of the atmosphere.

Budget Term	Shorthand	Description
$\partial_t A_P$		Local Eulerian PAPE tendency
$\frac{\gamma}{\overline{T}} T' Q_1'$	Q_p	Diabatic generation of PAPE
$-\frac{\gamma c_p}{\overline{T}} \left(\vec{V}_H' T' \right) \cdot \nabla \overline{T}$	B_c	Baroclinic conversion of mean APE to PAPE
$\frac{R}{\overline{p}} \omega' T'$	C_{pk}	Baroclinic conversion of PAPE to PKE

Table 1: Description of terms in the PAPE budget.

Note that if we had derived the PAPE budget using (2) instead of (3), we would get a very similar result. The third term of (16) would be of the form,

$$\overline{\omega' s'} = c_p \overline{\omega' T'} + g \overline{\omega' z'}.$$

The first term would represent the baroclinic conversion of PAPE to PKE, similar to the last term in (16). The second term is a vertical flux of geopotential that also represents a conversion of PAPE. It seems this should be related to the geopotential flux convergence term found in the PKE budget, but it is not clear how to directly relate these terms.

References

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