Eddy Vorticity and Enstrophy Budgets

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Motivation

Velocity and vorticity (i.e. rotation) offer interchangeable approaches to studying wave dynamics in a fluid. However, both velocity and vorticity are vector quantities, thus accounting for the full budget of either quantity can become rather complicated. To simplify this problem we can compress the information in the velocity vector into a scalar kinetic energy, which has a single budget equation that is useful for understanding the flow of energy in a system, as well as wave dynamics. We can make a similar simplification of the vorticity vector the obtain a scalar measure of rotational energy, known as enstrophy. Understanding fluid flows in terms of vorticity, instead of velocity, can yield some unique insights, and so the enstrophy budget provides a useful perspective. In particular, the eddy enstrophy budget can be a useful way to analyze how diabatic heating from things like convection and radiation affect wave dynamics. The potential vorticity and potential enstrophy budgets are also useful to consider, but these are not discussed here.

The following discussion outlines the derivation of the perturbation enstrophy budget. The term ”perturbation” indicates either anomalies relative to a temporal average, or high frequency fluctuations relative to a low frequency ”background” state that would be produced with a temporal filter.

The Isobaric Vorticity Budget

Consider the equations for horizontal momentum on isobaric surfaces,

$$\frac{\partial \vec{v}}{\partial t} + \vec{u} \cdot \nabla \vec{v} + f\hat{k} \times \vec{v} + \nabla_s \Phi = \vec{F}, \tag{1}$$

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where $\vec{v} = (u, v, 0)$ is the horizontal wind, $\vec{u} = (u, v, w)$ is the total wind, f is the coriolis parameter, and $\Phi = gz$ is the geopotential height above mean sea level. The vector $\vec{F}$ represents mechanical forcing.

We can define the hydrostatic relative vorticity as,

$$\vec{\zeta} = \nabla \times \vec{v} = -\hat{i}\frac{\partial v}{\partial p} + \hat{j}\frac{\partial u}{\partial p} + \hat{k}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]. \tag{2}$$

Using the following vector identity,

$$\vec{u} \times (\nabla \times \vec{v}) = \nabla\left(\frac{\vec{v} \cdot \vec{v}}{2}\right) - \vec{u} \cdot \nabla\vec{v} = -\vec{\zeta} \times \vec{u}, \tag{3}$$

we can rewrite (1) as

$$\frac{\partial \vec{v}}{\partial t} + \vec{\zeta} \times \vec{u} + \nabla\left(\frac{\vec{v} \cdot \vec{v}}{2}\right) + f\hat{k} \times \vec{v} + \nabla_s\Phi = \vec{F}. \tag{4}$$

Defining the absolute vorticity $\zeta_a = \vec{\zeta} + f$ and recognizing that $f\hat{k} \times \vec{v} = f\hat{k} \times \vec{u}$ we can simplify (4).

$$\frac{\partial \vec{v}}{\partial t} + \vec{\zeta_a} \times \vec{u} + \nabla\left(\frac{\vec{v} \cdot \vec{v}}{2}\right) + \nabla_s\Phi = \vec{F}. \tag{5}$$

We can now construct the vorticity equation by taking the curl of (5). Note that the curl of a divergence is zero so the third and forth terms immediately drops out.

$$\frac{\partial\left(\nabla \times \vec{v}\right)}{\partial t} + \nabla \times \left(\vec{\zeta_a} \times \vec{u}\right) = \nabla \times \vec{F}. \tag{6}$$

The second term can be expanded,

$$\nabla \times \left(\vec{\zeta_a} \times \vec{u}\right) = \left(\vec{u} \cdot \nabla\right)\vec{\zeta_a} - \left(\vec{\zeta_a} \cdot \nabla\right)\vec{u} + \vec{\zeta_a}\nabla \cdot \vec{u} - \vec{u}\left(\nabla \cdot \vec{\zeta_a}\right), \tag{7}$$

but since the total wind and absolute vorticity are non-divergent we can immediately simplify this to,

$$\nabla \times \left(\vec{\zeta_a} \times \vec{u}\right) = \left(\vec{u} \cdot \nabla\right)\vec{\zeta_a} - \left(\vec{\zeta_a} \cdot \nabla\right)\vec{u} \tag{8}$$

Making this substitution and recognizing that $\frac{\partial f}{\partial t} = 0$ allows us to rewrite (6) as,

$$\frac{\partial \vec{\zeta_a}}{\partial t} + \left(\vec{u} \cdot \nabla\right)\vec{\zeta_a} = \left(\vec{\zeta_a} \cdot \nabla\right)\vec{u} + \nabla \times \vec{F}, \tag{9}$$

which is the final isobaric vorticity equation. The first term on the right hand side contains the effects of stretching and tilting. Most often we are only interested in the vertical component of absolute vorticity (ζ_z), which can be isolated as,

$$\frac{\partial \zeta_z}{\partial t} + \vec{u} \cdot \nabla\zeta_z = \zeta_z\frac{\partial \omega}{\partial p} - \left(\hat{k} \cdot \nabla\omega \times \frac{\partial \vec{u}}{\partial p}\right) + \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}. \tag{10}$$

In a more computation-friendly notation we have,

$$\boldsymbol{\frac{\partial \zeta_z}{\partial t} + \vec{u} \cdot \nabla \zeta_z = \zeta_z \frac{\partial \omega}{\partial p} - \left(\frac{\partial v}{\partial p} \frac{\partial \omega}{\partial x} - \frac{\partial u}{\partial p} \frac{\partial \omega}{\partial y} \right) + \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}.} \tag{11}$$

The Eddy Vorticity Budget

The budget of eddy vorticity can be obtained from (11) by separating each variable into perturbations, indicated by a prime, and background states, indicated by an overbar. Note that the time derivative of barred quantities are not zero, as we want to allow for the barred quantities to represent low-frequency background states rather than a strict temporal average. Starting with the advection terms, separating the perturbations gives,

$$\left(\overline{\vec{u}} + \vec{u}' \right) \cdot \nabla \left(\overline{\zeta_z} + \zeta_z' \right) = \overline{\vec{u}} \cdot \nabla \overline{\zeta_z} + \overline{\vec{u}} \cdot \nabla \zeta_z' + \vec{u}' \cdot \nabla \overline{\zeta_z} + \vec{u}' \cdot \nabla \zeta_z'. \tag{12}$$

Similarly, we can expand the vortex stretching term,

$$\left(\overline{\zeta_z} + \zeta_z' \right) \frac{\partial \left(\overline{\omega} + \omega' \right)}{\partial p} = \overline{\zeta_z} \frac{\partial \overline{\omega}}{\partial p} + \overline{\zeta_z} \frac{\partial \omega'}{\partial p} + \zeta_z' \frac{\partial \overline{\omega}}{\partial p} + \zeta_z' \frac{\partial \omega'}{\partial p} \tag{13}$$

Calculating the "background state" of all terms leaves the background vorticity balance,

$$\begin{aligned} \frac{\partial \overline{\zeta_z}}{\partial t} + \overline{\vec{u}} \cdot \nabla \overline{\zeta_z} + \overline{\vec{u}' \cdot \nabla \zeta_z'} =& \overline{\zeta_z} \frac{\partial \overline{\omega}}{\partial p} + \overline{\zeta_z' \frac{\partial \omega'}{\partial p}} \\ & - \left(\frac{\partial \overline{v}}{\partial p} \frac{\partial \overline{\omega}}{\partial x} + \overline{\frac{\partial v'}{\partial p} \frac{\partial \omega'}{\partial x}} - \frac{\partial \overline{u}}{\partial p} \frac{\partial \overline{\omega}}{\partial y} - \overline{\frac{\partial u'}{\partial p} \frac{\partial \omega'}{\partial y}} \right) \\ & + \frac{\partial \overline{F_y}}{\partial x} - \frac{\partial \overline{F_x}}{\partial y}. \end{aligned} \tag{14}$$

In some cases this operation will simply be the time mean, whereas in other cases it may be more appropriate to use a low-pass filter. In the case of a low-pass filter where the barred quantities retain a time dimension, we have omitted some additional non-linear terms above. In practice these non-linear terms are hard to interpret, and often have very small magnitudes, So unless a special need arises to consider these terms, you shouldn't feel bad about neglecting them.

Subtracting (14) from (11) gives the budget of eddy vorticity,

$$
\begin{aligned}
\frac{\partial \zeta_z'}{\partial t} =& - \vec{u}' \cdot \nabla \overline{\zeta_z} - \overline{\vec{u}} \cdot \nabla \zeta_z' - \left(\vec{u}' \cdot \nabla \zeta_z'\right)' \\
& + \zeta_z' \frac{\partial \overline{\omega}}{\partial p} + \overline{\zeta_z} \frac{\partial \omega'}{\partial p} + \left(\zeta_z' \frac{\partial \omega'}{\partial p}\right)' \\
& - \left(\frac{\partial v'}{\partial p} \frac{\partial \overline{\omega}}{\partial x} + \frac{\partial \overline{v}}{\partial p} \frac{\partial \omega'}{\partial x} + \left(\frac{\partial v'}{\partial p} \frac{\partial \omega'}{\partial x} \right)' - \frac{\partial u'}{\partial p} \frac{\partial \overline{\omega}}{\partial y} - \frac{\partial \overline{u}}{\partial p} \frac{\partial \omega'}{\partial y} - \left(\frac{\partial u'}{\partial p} \frac{\partial \omega'}{\partial y} \right)' \right) \\
& + \frac{\partial F_y'}{\partial x} - \frac{\partial F_x'}{\partial y}.
\end{aligned}
\tag{15}
$$

The Eddy Enstrophy Budget

Enstrophy is a measure of rotational energy defined as square of the vorticity divided by 2,

$$
\xi = \frac{\zeta^2}{2}. \tag{16}
$$

Thus the budget of the vertical component of eddy enstrophy can be obtained by multiplying (17) by the eddy vorticity, ζ'.

$$
\begin{aligned}
\frac{\partial \xi_z'}{\partial t} =& - \zeta_z' \vec{u}' \cdot \nabla \overline{\zeta_z} - \overline{\vec{u}} \cdot \nabla \xi_z' - \zeta_z' \left(\vec{u}' \cdot \nabla \zeta_z'\right)' \\
& + 2\xi_z' \frac{\partial \overline{\omega}}{\partial p} + \zeta_z' \overline{\zeta_z} \frac{\partial \omega'}{\partial p} + \zeta_z' \left(\zeta_z' \frac{\partial \omega'}{\partial p}\right)' \\
& - \zeta_z' \left(\frac{\partial v'}{\partial p} \frac{\partial \overline{\omega}}{\partial x} + \frac{\partial \overline{v}}{\partial p} \frac{\partial \omega'}{\partial x} + \left(\frac{\partial v'}{\partial p} \frac{\partial \omega'}{\partial x} \right)' - \frac{\partial u'}{\partial p} \frac{\partial \overline{\omega}}{\partial y} - \frac{\partial \overline{u}}{\partial p} \frac{\partial \omega'}{\partial y} - \left(\frac{\partial u'}{\partial p} \frac{\partial \omega'}{\partial y} \right)' \right) \\
& + \zeta_z' \frac{\partial F_y'}{\partial x} - \zeta_z' \frac{\partial F_x'}{\partial y}.
\end{aligned}
\tag{17}
$$

The first advection term on the right hand side represents the exchange of enstrophy between the background flow and transient eddies, similar to the barotropic energy conversion term found in the eddy kinetic energy budget. This term is positive when the eddy vorticity flux is directed down the mean absolute vorticity gradient. The second term represents advection of enstrophy perturbations by the background flow. The third non-linear term in (17) represents perturbation enstrophy advection by the transient eddies.

The second line includes two terms that describe the generation of enstrophy by vortex stretching, which are important for considering the effects of diabatic heating. The second term involving the vertical derivative of vertical velocity perturbations generally dominates over the first. The third non-linear term is generally small.

The third line includes terms that describe the effects of tilting. All of these terms are generally small compared to the advection and vortex stretching terms. The final terms describe enstrophy generation or destruction by mechanical forcing such as surface friction and convective momentum transport.