# VI.—*On Vortex Motion.* By Sir W. THOMSON.

(Read 29th April 1867.)

(§§ 1–59 recast and augmented 28th August to 12th November 1868.)

1. The mathematical work of the present paper has been performed to illustrate the hypothesis, that space is continuously occupied by an incompressible frictionless liquid acted on by no force, and that material phenomena of every kind depend solely on motions created in this liquid. But I take, in the first place, as subject of investigation, a finite mass of incompressible frictionless* fluid completely enclosed in a rigid fixed boundary.

2. The containing vessel may be either *simply* or *multiply continuous*.† And I shall frequently consider solids surrounded by the liquid, which also may be either simply or multiply continuous. It will not be necessary to exclude the supposition that any such solid may touch the outer boundary over some finite area, in which case it is *not* surrounded by the liquid; but each such solid, whether surrounded by the liquid or not, and whether moveable or fixed, must be considered as a part of the whole boundary of the liquid.

3. Let the whole fluid be given at rest, and let no force, except pressure from the containing vessel, or from the surfaces of solids immersed in it, ever act on any part of it. Let there be any number of solids, perfectly incompressible, and of the same density as the fluid; but either perfectly rigid, or more or less flexible, with perfect or imperfect elasticity. Some of these may at times be supposed to lose rigidity, and become perfectly liquid; and portions of the liquid may be supposed to acquire rigidity, and thus to constitute solids. Let the solids act on one another with any forces, pressures, frictions, or mutual distant actions, subject only to the law of "action and reaction." Let motions originate among them and in the liquid, either by the natural mutual actions of the solids or by the arbitrary application of forces to them during some limited time. It is of no consequence to us whether these forces have reactions on matter outside the containing vessel, so that they might be called "natural forces" in the present state of science (which admits action and reaction at a distance); or are applied arbitrarily by supernatural action without reaction. To avoid circumlocution,

* A frictionless fluid is defined as a mass continuously occupying space, whose contiguous portions press on one another everywhere exactly in the direction perpendicular to the surface separating them.

† HELMHOLTZ—*Ueber Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen:* Crelle (1858); translated by TAIT in Phil. Mag. 1867, i. RIEMANN—*Lehrsätze aus der Analysis situs, &c.* Crelle (1857). See also § 58, below.

and, at the same time, to conform to a common usage, we shall call them *impressed forces*.

4. From the homogeneousness as to density of the contents of the fixed bounding vessel, it follows that the centre of inertia of the whole system of liquid and solids immersed in it remains at rest; in other words, the integral momentum of the motion is zero. Hence (THOMSON and TAIT'S "Natural Philosophy," § 297) the time integral of the sum of the components of *pressure on the containing vessel*, parallel to any fixed line, is equal to the time-integral of the sum of the components of *impressed forces* parallel to the same line. This equality exists, of course, at each instant during the action of the impressed forces, and continues to exist for the constant values of their time integrals, after they have ceased. Thus, in the subsequent motion of the solids, and of the fluids compelled to yield to them, whatever pressure may come to act on the containing vessel, whether from the fluid or from some of the solids coming in contact with it, the components of this pressure, parallel to any fixed line, summed for every element of the inner surface of the vessel, must vanish for every interval of time during which no impressed forces act. If, for example, one of the solids strikes the containing vessel, there will be an impulsive pressure of the fluid over all the rest of the fixed containing surface, having the sum of its components parallel to any line, equal and contrary* to the corresponding component of the impulsive pressure of the solid on the part of this surface which it strikes [see § 8, and consider oblique impulse of an inner moving solid, on the fixed solid spherical boundary]. *But, after the impressed forces cease to act, and as long as the containing vessel is not touched by any of the solids, the integral amount of the component of fluid pressure on it, parallel to any line, vanishes.*

5. If now forces be applied to stop the whole motion of fluid and solids [as (§ 62) is done, if the solids are brought to rest by forces applied to themselves only], the time integrals of the sums of the components of these forces, parallel to any stated lines, *may or may not in general be equal and contrary* to the time integrals of the corresponding sums of components of the initiating impressed forces (§ 3). But we shall see (§§ 19, 21), that *if the containing vessel be infinitely large, and all of the moving solids be infinitely distant from it during the whole motion*, there must be not merely the equality in question between the time integrals of the components in contrary directions of the initiating and stopping impressed forces, but there must be (§ 21) *completely equilibrating opposition between the two systems.*

6. To avoid circumlocution, henceforth I shall use the unqualified term *impulse* to signify a system of impulsive forces, to be dealt with as if acting on a rigid body. Thus the most general impulse may be reduced to an impulsive force, and couple

* I shall use the word *contrary* to designate merely directional opposition; and reserve the unqualified word *opposite*, to signify *contrary and in one line.*

in plane perpendicular to it, according to POINSOT; or to two impulsive forces in lines not meeting, according to his predecessors. Further, I shall designate by *the impulse of the motion at any instant*, in our present subject, the system of impulsive forces on the moveable solids which would generate it from rest; or any other system which would be equivalent to that one if the solids were all rigid and rigidly connected with one another, as, for instance, the POINSOT resultant impulsive force and minimum couple. The line of this resultant impulsive force will be called the *resultant axis of the motion*, and the moment of the minimum couple (whose plane is perpendicular to this line) will be called the *rotational moment* of the motion.

7. But, having thus defined the terms I intend to use, I must, to warn against errors that might be fallen into, remark that the momentum of the whole motions of solids and liquid is *not* equal to what I have defined as *the impulse*, but (§ 4) is equal to zero; being the force-resultant of "the impulse" and the impulsive pressure exerted on the liquid by the containing vessel during the generation of the motion: and that the moment of momentum of the whole motion round the centre of inertia of the contents of the vessel is *not* equal to the *rotational moment*, as I have defined it, but is equal to the moment of the couple constituted by "the impulse" and the impulsive pressure of the containing vessel on the liquid. It must be borne in mind that however large, and however distant all round from the moveable solids, the containing vessel may be, it exercises a finite influence on the momentum and moment of momentum of the whole motion within it. But if it is infinitely large, and infinitely distant all round from the solids, it does so by infinitely slow motion through an infinitely large mass of fluid, and exercises no finite influence on the finite motion of the solids or of the neighbouring fluid. This will be readily understood, if for an instant we suppose the rigid containing vessel to be not fixed, but quite free to move as a rigid body without mass. The momentum of the whole motion will then be not zero, but exactly equal to the force-resultant of the impulse on the solids; and the moment of momentum of the whole motion round the centre of inertia will be precisely equal to the resultant impulsive couple found by transposing the constituent impulsive forces to this point after the manner of POINSOT. But the finite motion of the immersed solids, and of the fluid in their neighbourhood which we shall call the *field of motion*, will not be altered by any finite difference, whether the containing vessel be held fixed or left free, provided it be infinitely distant from them all round. It is, therefore, essentially indifferent whether we keep it fixed or let it be free. The former supposition is more convenient in some respects, the latter in others; but it would be inconvenient to leave any ambiguity, and I shall adhere (§ 1) to the former in all that follows.

8. To further illustrate the impulse of the motion, and its resultant impulsive force and couple, according to the previous definitions, as distinguished from

the momentum, and the moment of momentum, of the whole contents of the vessel, let the vessel be spherical. Its impulsive pressure on the liquid will always be reducible to a single resultant in a line through its centre, which (§ 4) will be equal and contrary to the force-resultant of "the impulse;" and, therefore, with it will constitute in general a couple. The resultant, of this couple and the couple-resultant of the impulse, will be equal to the moment of momentum of the whole motion round the centre of the sphere (which is the centre of inertia). But if the vessel be infinitely large, and infinitely distant all round from the moveable solids, the moment of momentum of the whole motion is irrelevant; and what is essentially important, is the impulse and its force and couple-resultants, as defined above.

9. The following way of stating (§§ 10, 12), and proving (§§ 11—15), a fundamental proposition in fluid motion will be useful to us for the theory of the impulse, whether of the moveable solids we have hitherto considered or of vortices.

10. The moment of momentum of every spherical portion of a liquid mass in motion, relatively to the centre of the sphere, is always zero, if it is so at any one instant for every spherical portion of the same mass.

11. To prove this, it is first to be remarked, that the moment of momentum of that part of the liquid which at any instant occupies a certain fixed spherical space can experience no change, at that instant (or its *rate* of change vanishes at that instant), because the fluid pressure on it (§ 1), being perpendicular to its surface, is everywhere precisely towards its centre. Hence, if the moment of momentum of the matter in the fixed spherical space varies, it must be by the moment of momentum of the matter which enters it not balancing exactly that of the matter which leaves it. We shall see later (§§ 20, 17, 18) that this balancing is vitiated by the entry of either a moving solid, or of some of the liquid, if any there is, of which spherical portions possess moment of momentum, into the fixed spherical space; but it is perfect under the condition of § 10, as will be proved in § 15.

12. First, I shall prove the following purely mathematical lemmas; using the ordinary notation $u$, $v$, $w$ for the components of fluid velocity at any point $(x, y, z)$.

Lemma (1.) The condition (last clause) of § 10 requires that $u\,dx + v\,dy + w\,dz$ be a complete differential,* at whatever instant and through whatever part of the fluid the condition holds.

Lemma (2.) If $u\,dx + v\,dy + w\,dz$ be a complete differential of a single valued function of $x$, $y$, $z$, through any finite space of the fluid, at any instant, the condition of § 10 holds through that space at that instant.

* This proposition was, I believe, first proved by STOKES in his paper "On the Friction of Fluids in Motion, and the Equilibrium and Motion of Elastic Solids."—"Cambridge Philosophical Transactions," 14th April 1845.

13. The following is STOKES' proof of Lemma (1):—First, for any motion whatever, whether subject to the condition of § 10 or not, let L be the component moment of momentum round OX of an infinitesimal sphere with its centre at O. Denoting by $\iiint$ integration through this space, we have

$$L = \iiint (wy - vz)\, dx\, dy\, dz \quad . \quad . \quad . \quad (1).$$

Now let $\left(\frac{dw}{dx}\right)_0$, $\left(\frac{dw}{dy}\right)_0$, &c. denote the values at O of the differential coefficients. We have, by MACLAURIN's theorem,

$$w = x\left(\frac{dw}{dx}\right)_0 + y\left(\frac{dw}{dy}\right)_0 + z\left(\frac{dw}{dz}\right)_0 ,$$

and so for $v$. Hence, remembering that $\left(\frac{dw}{dx}\right)_0$, &c. are constants for the space through which the integration is performed, we have

$$\iiint dx\, dy\, dz\, wy = \left(\frac{dw}{dx}\right)_0 \iiint xy\, dx\, dy\, dz + \left(\frac{dw}{dy}\right)_0 \iiint y^2 dx\, dy\, dz + \left(\frac{dw}{dz}\right)_0 \iiint zy\, dx\, dy\, dz .$$

The first and third of the triple integrals vanish, because every diameter of a homogeneous sphere is a principal axis; and if A denote moment of momentum of the spherical volume round its centre, we have for the second

$$\iiint y^2\, dx\, dy\, dz = \tfrac{1}{2} A .$$

Dealing similarly with $vz$ in the expression for L, we find

$$L = \tfrac{1}{2} A \left[\left(\frac{dw}{dy}\right)_0 - \left(\frac{dv}{dz}\right)_0\right] \quad . \quad . \quad . \quad (2).$$

But L must be zero according to the condition of § 10; and, therefore, as the centre of the infinitesimal sphere now considered may be taken at any point of space through which this condition holds at any instant, we must have, throughout that space,

$$\left.\begin{aligned} \frac{dw}{dy} - \frac{dv}{dz} &= 0 \\ \frac{du}{dz} - \frac{dw}{dx} &= 0 \\ \frac{dv}{dx} - \frac{du}{dy} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3);$$

and similarly

which proves Lemma (1.)

14. To prove Lemma (2.), let

$$u = \frac{d\phi}{dx}, \quad v = \frac{d\phi}{dy}, \quad w = \frac{d\phi}{dz} \quad . \quad . \quad . \quad . \quad . \quad (4);$$

and let L denote the component moment of momentum round OX, through any spherical space with O in centre. We have [(1) of § 13],

$$\mathrm{L} = \iiint dx\, dy\, dz\, (wy - vz) \quad . \quad . \quad . \quad . \quad . \quad (5),$$

$\iint$ denoting integration throughout this space (not now infinitesimal). But by (4)

$$yw - vz = \left(y \frac{d}{dz} - z \frac{d}{dy}\right) \varphi = \frac{d\varphi}{d\psi} \quad . \quad . \quad (6);$$

if $\frac{d}{d\psi}$ denote differentiation with reference to $\psi$, in the system of co-ordinate $x$, $\rho$, $\psi$, such that

$$y = \varrho \cos \psi\,,\ z = \varrho \sin \psi \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Hence, transforming (5) to this system of co-ordinates, we have

$$\mathrm{L} = \iiint dx\, d\varrho\, \varrho d\psi \frac{d\varphi}{d\psi} \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Now, as the whole space is spherical, with the origin of co-ordinates in its centre, we may divide it into infinitesimal circular rings with OX for axis, having each for normal section an infinitesimal rectangle with $dx$ and $d\rho$ for sides. Integrating first through one of these rings, we have

$$dx\, d\varrho\, \varrho \int_0^{2\pi} \frac{d\varphi}{d\psi}\, d\psi\,,$$

which vanishes, because $\varphi$ is a single-valued function of the co-ordinates. Hence $\mathrm{L} = 0$, which proves Lemma (2.).

15. Returning now to the dynamical proposition, stated at the conclusion of § 11; for the promised proof, let R denote the radial component velocity of the fluid across any element, $d\sigma$, of the spherical surface, situated at $(x, y, z)$; and let $u, v, w$ be the three components of the resultant velocity at this point; so that

$$\mathrm{R} = u\frac{x}{r} + v\frac{y}{r} + w\frac{z}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

The volume of fluid leaving the hollow spherical space across $d\sigma$ in an infinitesimal time, $dt$ is $\mathrm{R}d\sigma\,.\,dt$, and the moment of momentum of this moving mass round the centre has, for component round OX,

$$(wy - vz)\,\mathrm{R}d\sigma\, dt.$$

Hence, if L denote the component of the moment of momentum of the whole, mass within the spherical surface at any instant, $t$, we have (§ 11),

$$\frac{d\mathrm{L}}{dt} = \iint (wy - vz)\,\mathrm{R}\, d\sigma, \quad . \quad . \quad . \quad . \quad . \quad (10).$$

Now, using Lemma (1.) of § 12, and the notation of § 14, we have

$$wy - vz = \frac{d\varphi}{d\psi}\,,$$

and, by (9),

$$R = \frac{d\varphi}{dr}$$

where $\frac{d}{dr}$ denotes rate of variation per unit length perpendicular to the spherical surface, that is differentiation with reference to $r$, the other two co-ordinates being directional relatively to the centre. Hence, using ordinary polar co-ordinates, $r$, $\theta$, $\psi$, we have

$$\frac{dL}{dt} = r^2 \iint \frac{d\varphi}{dr} \frac{d\varphi}{d\psi} \sin\theta \, d\theta d\psi \quad . \quad . \quad . \quad . \quad (11).$$

But the "equation of continuity" for an incompressible liquid (being

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0),$$

gives* $\nabla^2 \phi = 0$, for every point within the spherical space; and therefore [THOMSON & TAIT, App. B]

$$\varphi = S_0 + S_1 r + S_2 r^2 + \&c. \quad . \quad . \quad . \quad . \quad (12).$$

a converging series, where $S_0$ denotes a constant, and $S_1$, $S_2$, &c., surface harmonics of the orders indicated.

Hence

$$R = \frac{d\varphi}{dr} = S_1 + 2r S_2 + 3 r^2 S_3 + \&c. \quad . \quad . \quad (13).$$

And it is clear from the synthesis of the most general surface harmonic, by zonal, sectional, and tesseral harmonics [THOMSON & TAIT, §781], that $\frac{dS_i}{d\psi}$ is a surface harmonic of the same order as $S_i$:† from which [THOMSON & TAIT, App. B (16)], it follows that,

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* By $\nabla^2$ we shall always understand $\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$.

† This follows, of course, from the known analytical theorem that the operations $\nabla^2$ and $\left(y \frac{d}{dz} - z \frac{d}{dy}\right)$ are commutative, which is proved thus:—

By differentiation we have

$$\frac{d^2\left(y \frac{d\varphi}{dz}\right)}{dy^2} = y \frac{d^2}{dy^2} \frac{d\varphi}{dz} + 2 \frac{d}{dy} \frac{d\varphi}{dz};$$

and therefore, since $\frac{d}{dy} \frac{d\varphi}{dz} = \frac{d}{dz} \frac{d\varphi}{dy}$,

$$\nabla^2 \left(y \frac{d\varphi}{dz} - z \frac{d\varphi}{dy}\right) = y \nabla^2 \left(\frac{d\varphi}{dz}\right) - z \nabla^2 \left(\frac{d\varphi}{dy}\right) = \left(y \frac{d}{dz} - z \frac{d}{dy}\right) \nabla^2 \varphi$$

or

$$\nabla^2 \left(y \frac{d}{dz} - z \frac{d}{dy}\right) \varphi = \left(y \frac{d}{dz} - z \frac{d}{dy}\right) \nabla^2 \varphi,$$

$\varphi$ being any function whatever. Hence, if $\nabla^2 \varphi = 0$ we have

$$\nabla^2 \left(y \frac{d\varphi}{dz} - z \frac{d\varphi}{dy}\right) = 0 \cdot$$

$$\iint S_i \frac{dS_{i'}}{d\psi} \sin\theta \, d\theta \, d\psi = 0 \, ,$$

except when $i' = i$. But this is true also when $i' = i$ because

$$S_i \frac{dS_i}{d\psi} = \tfrac{1}{2} \, d \, \frac{(S_i^2)}{d\psi} \, ,$$

and therefore, as in § 14, the integration for $\psi$, from $\psi = 0$ to $\psi = 2\pi$ gives zero. Hence (11) gives

$$\frac{dL}{dt} = 0 \, ,$$

This and § 11 establish § 10.

16. Lemma (1) of § 11, and § 10 now proved, show that in any motion whatever of an incompressible liquid, whether with solids immersed in it or not, $udx + vdy + wdz$ is always a complete differential through any portion of the fluid, for which it is a complete differential at any instant, to whatever shape and position of space this portion may be brought in the course of the motion. This is the ordinary statement of the fundamental proposition of fluid motion referred to in § 9, which was first discovered by LAGRANGE. (For another proof see § 60.) I have given the preceding demonstration, not so much because it is useful to look at mathematical structures from many different points of view, but (§ 19) because the dynamical considerations and the formulæ I have used are immediately available for establishing the theory of the impulse (§§ 3 . . . 8), of which a fundamental proposition was stated above (§ 5). To prove this proposition (in § 19) I now proceed.

17. Imagine any spherical surfaces to be described round a moveable solid or solids immersed in a liquid. The surrounding fluid can only press (§ 1) perpendicularly; and therefore when any motion is (§ 3) generated by impulsive forces applied to the solids, the moment round any diameter of the momentum of the matter within the spherical surface at the first instant, must be exactly equal to the moment of those impulsive forces round this line. And the moment round this line, of the momentum of the matter in the space between any two concentric spherical surfaces is zero, provided neither cuts any solid, and provided that, if there are any solids in this space, no impulse acts on them.

18. Hence, considering what we have defined as "the impulse of the motion," (§ 6), we see that its moment round any line is equal to the moment of momentum round the same line, of all the motion within any spherical surface having its centre in this line, and enclosing all the matter to which any constituent of the impulse is applied. This will still hold, though there are other solids not in the neighbourhood, and impulses are applied to them: provided the moments of momentum of those only which are within S are taken into account, and provided none of them is cut by S.

19. The statements of § 11, regarding fluid occupying at any instant a fixed spherical surface, are applicable without change to the fluids and solids occupying

the space bounded by S, because of our present condition, that no solid is cut by S. Hence every statement and formula of § 15, as far as equation (11), may be now applied to the matter within S; but instead of (12) we now have [THOMSON & TAIT, § 736], if we denote by $T_1$, $T_2$, &c., another set of surface spherical harmonics,

$$\left.\begin{aligned}\varphi = S_0 + S_1 r \quad + S_2 r^2 \quad + \&c. \\ + T_1 r^{-2} + T_2 r^{-3} + \&c.\end{aligned}\right\} \quad . \quad . \quad . \quad (14).*$$

for all space between the greatest and smallest spherical surface concentric with S, and having no solids in it, because through all this space, § 16, and the equation of continuity prove that $\nabla^2 \phi = 0$. Hence, instead of (13), we now have

$$\left.\begin{aligned}R = \frac{d\varphi}{dr} = S_1 + 2r\,S_2 + 3r^2\,S_3, \&c. \\ - \frac{2}{r^3} T_1 - \frac{3}{r^4} T_2 - \frac{4}{r^5} T_3 + \&c.\end{aligned}\right\} \quad . \quad (15).$$

Hence finally

$$\frac{dL}{dt} = \sum_{i=0}^{i=\infty} \iint \left[ i\,S_i \frac{d\,T_i}{d\psi} - (i+1)\,T_i \frac{dS_i}{d\psi} \right] \sin\theta d\theta d\psi \quad . \quad (16).$$

Now if, as assumed in § 5, neither any moveable solids, nor any part of the boundary exist within any finite distance of S all round; $S_1$, $S_2$, &c., must each be infinitely small: and therefore (16) gives $\frac{dL}{dt} = 0$. This proves the proposition asserted in § 5: because a system of forces cannot have zero moment round every line drawn through any finite portion of space, without having force-resultant and couple-resultant each equal to zero.

20. As the rigidity of the solids has not been taken into account, all or any of them may be liquefied (§ 3) without violating the demonstration of § 19. To save circumlocutions, I now define a *vortex* as a portion of fluid having any motion that it could not acquire by fluid pressure transmitted through itself from its boundary. Often, merely for brevity, I shall use the expression *a body* to denote either a solid or a vortex, or a group of solids or vortices.

21. The proposition thus proved may be now stated in terms of the definitions of § 6, which were not used in § 5, and so becomes simply this:—*The impulse of the motion of a solid or group of solids or vortices and the surrounding liquid remains constant as long as no disturbance is suffered from the influence of other solids or vortices, or of the containing vessel.*

This implies, of course (§ 6), that the magnitudes of the force-resultant and the rotational moment of the impulse remain constant, and the position of its axis invariable.

* There is no term $\frac{T_0}{r}$, because this would give, in the integral of flow across the whole spherical surface, a finite amount of flow out of or into the space within, implying a generation or destruction of matter.

22. In POINSOT'S system of the statics of a rigid body we may pass from the resultant force and couple along and round the central axis to an equal resultant force along the parallel line through any point, and a greater couple the resultant of the former (or minimum) couple, and a couple in the plane of the two parallels, having its moment equal to the product of their distance into the resultant force. So we may pass from the force-resultant and rotational moment of the impulse along and round its axis, to an equal force-resultant and greater moment of impulse, by transferring the former to any point, Q, not in the axis (§ 6) of the motion. This greater moment is (§ 18) equal to the moment of momentum round the point Q, of the motion within any spherical surface described from Q as centre, which encloses all the vortices or moving solids.

23. Hence a group of solids or vortices which always keep within a spherical surface of finite radius, or a single body, moving in an infinite liquid, can have no permanent average motion of translation in any direction oblique to the direction of the force-resultant of the impulse, if there is a finite force-resultant. For the matter within a finite spherical surface enclosing the moving bodies or body, cannot have moment of momentum round the centre increasing to infinity.

24. But there may be motion of translation when the force-resultant of the impulse vanishes; and there will be, for example, in the case of a solid, shaped like the screw-propeller of a steamer, immersed in an infinite homogeneous liquid, and set in motion by a couple in a plane perpendicular to the axis of the screw.

25. And when the force-resultant of the impulse does *not* vanish, there may be no motion of translation, or there may even be translation in the direction opposite to it. Thus, for example, a rigid ring, with cyclic motion, established (§ 63) through it, will, if left at rest, remain at rest. And if at any time urged by an impulse in either direction in the line of the force-resultant of the impulse of the cyclic motion, it will commence and continue moving with an average motion of translation in that direction; a motion which will be uniform, and the same as if there were no cyclic motion, when the ring is symmetrical. If the translatory impulse is contrary to the cyclic impulse, but less in magnitude, the translation will be contrary to the whole force-resultant impulse.

If the translatory impulse is equal and opposite to the cyclic impulse, there will be translation with zero force-resultant impulse—another example of what is asserted in § 24. In this case, if the ring is plane and symmetrical, or of any other shape such that the cyclic motion (which, to fix ideas, we have supposed given first, with the ring at rest,) must have had only a force-resultant, and no rotational moment, we have a solid moving with a uniform motion of translation through a fluid, and both force and couple resultant of the whole impulse zero.

26. From §§ 21 and 4, we see that, however long the time of application of the impressed forces may be—provided only that, during the whole of it, the

solid or group of solids has been at an infinite distance from all other solids and from the containing vessel—the time integrals of the impressed forces parallel to three fixed axes, and of their moments round these lines, are equal to the six corresponding components of "the impulse" (§ 6).

27. If two groups, at first so far asunder as to exercise no sensible influence on one another, come together, the "impulse" of the whole system remains unchanged by any disturbance each may experience from the other, whether by impacts of the solids, or through motion and pressure of the surrounding fluid; and (§ 6) it is always reducible to the force-resultant along the central axis, and the minimum couple-resultant, of the two impulses reckoned as if applied to one rigid body. The same holds, of course, if one group separates into two so distant as to no longer exert any sensible influence on one another.

28. Hence whatever is lost of impulse perpendicular to a fixed plane, or of component rotational movement round a fixed line, by one group through collision with another, is gained by the other.

29. Two of the moveable solids, or two groups, will be said to be *in collision* when, having been so far asunder as not to disturb one another's motions sensibly, they are so near as to do so. This disturbance will generally be supposed to be through fluid pressure only, but impacts of solids on solids may take place during a collision.

30. We are now prepared to investigate (§§ 30, 31, 32) the influence of a fixed solid on the impulse of a moveable solid, or of a vortex, or of a group of solids or vortices, passing near it, thus—If during such collisions or separations as are considered in §§ 27, 28, forces are impressed on any one or more of the solids, their alteration of the whole impulse is (§ 26) to be reckoned by adding to each of its rectangular components the time integral of the corresponding component of these impressed forces. Now, let us suppose such forces to be impressed on any one of the moveable solids as shall keep it at rest. These forces are zero as long as no moving solid is within a finite distance. But if a moving solid or vortex, or group of solids or vortices, passes near the fixed solid, the change of pressure due to the motion of the fluid will tend to move it, and the impression of force on it becomes necessary to keep it fixed. Let $d\sigma$ be an element of its surface; $(x, y, z)$, the co-ordinates of the centre of this element; $\alpha, \beta, \gamma$ the inclinations of the normal at $(x, y, z)$ to the three rectangular axes; and $p$ the fluid pressure at time $t$, and point $(x, y, z)$. The six components of force and couple required to hold the body fixed at time $t$, are

$$\left.\begin{aligned}&\iint d\sigma \,.\, \cos\alpha \,.\, p\,,\ \iint d\sigma \,.\, \cos\beta \,.\, p\,,\ \iint d\sigma \,.\, \cos\gamma \,.\, p\,;\\ &\iint d\sigma\,(y\cos\gamma - z\cos\beta)p\,,\ \iint d\sigma(z\cos\alpha - x\cos\gamma)p\,,\ \iint d\sigma(x\cos\beta - y\cos x)p\,,\end{aligned}\right\} \quad (1).$$

If in these expressions we substitute

$$\int p dt \qquad \ldots \ldots \qquad (2).$$

in place of $p$ ($\int dt$ denoting a time integral from any era of reckoning before the disturbance became sensible, up to time $t$, which may be any instant during the collision, or after it is finished), we have the changes in the corresponding components of the impulse up to time $t$, provided there has been no impact of moveable solid on the fixed solid.

31. Let now the "velocity potential" (as we shall call it, in conformity with a German usage which has been adopted by HELMHOLTZ,) be denoted by $\phi$; that is (§ 16), let $\phi$ be such a function of $(x, y, z, t)$ that

$$u = \frac{d\varphi}{dx}, \; v = \frac{d\varphi}{dy}, \; w = \frac{d\varphi}{dz} \quad . \quad . \quad . \quad (3).$$

and let $\dot{\phi}$ $\left(\text{or } \frac{d\varphi}{dt}\right)$ denote its rate of variation per unit of time at any instant $t$, for the point $(x, y, z)$ regarded as fixed.

Also, let $q$ denote the resultant fluid velocity, so that

$$q^2 = u^2 + v^2 + w^2 = \frac{d\varphi^2}{dx^2} + \frac{d\varphi^2}{dy^2} + \frac{d\varphi^2}{dz^2} \quad . \quad . \quad (4).$$

The ordinary hydro-dynamical formula gives

$$p = \Pi - \dot{\varphi} - \tfrac{1}{2} q^2 \quad . \quad . \quad . \quad . \quad (5);$$

where $\Pi$ denotes the constant pressure in all sensibly quiescent parts of the fluid.

32. The constant term $\Pi$ disappears from $p$ in each of the integrals (1) of § 30, because a solid is equilibrated by equal pressure around. And in the time integral (2), we have

$$\int \dot{\varphi}\, dt = \varphi \quad . \quad . \quad . \quad . \quad . \quad (6);$$

and therefere if (XYZ) (LMN) denote the changes in the force-and couple-components of the impulse produced by the collision up to time $t$, we have

$$\left.\begin{aligned} \mathrm{X} &= -\iint d\sigma \cos\alpha \,(\varphi + \tfrac{1}{2}\int q^2\, dt),\ \mathrm{Y} = \&\text{c.}, \mathrm{Z} = \&\text{c.}, \\ \mathrm{L} &= -\iint d\sigma \,(y\cos\gamma - z\cos\beta)\,(\varphi + \tfrac{1}{2}\int q^2\, dt), \mathrm{M} = \&\text{c.}, \mathrm{N} = \&\text{c.}, \end{aligned}\right\} \quad . \quad . \quad . \quad (7).$$

But because the fluid is quiescent in the neighbourhood of the fixed body when the moving body or group of bodies is infinitely distant from it; it follows that before the commencement and after the end of the collision we have $\phi = 0$ at every point of the surface of the fixed body. Hence, for every value of $t$ representing a time after the completion of the collision, the preceding expressions become

$$\left.\begin{aligned} \mathrm{X} &= -\tfrac{1}{2}\iint d\sigma \cos\alpha \int q^2\, dt,\ \mathrm{Y} = \&\text{c.}, \mathrm{Z} = \&\text{c.}, \\ \mathrm{L} &= -\tfrac{1}{2}\iint d\sigma (y\cos\gamma - z\cos\beta) \int q^2\, dt, \mathrm{M} = \&\text{c.}, \mathrm{N} = \&\text{c.}, \end{aligned}\right\} \quad . \quad . \quad (8);$$

which express that *the integral change of impulse experienced by a body or group of bodies, in passing beside a fixed body without striking it, may be regarded as a* *system of impulsive attractions towards the latter, everywhere in the direction of the normal, and amounting to* $\frac{1}{2}\int q^2 dt$ *per unit of area.* But it must not be forgotten that the term $\phi$ in the expression [§ 31 (5)] for $p$ produces, as shown in § 30 (1), an influence *during the collision,* the integral effect of which only disappears from the expression [§ 32 (7)] for the impulse *after the collision is completed;* that is (§ 29) after the moving system has passed away so far as to leave no sensible fluid motion in the neighbourhood of the fixed body.

33. Hence, and from § 23, we see that when there is no impact of moving solid against the fixed body, and when the moving solid or group of solids passes altogether on one side of the fixed body, the direction of the translation will be deflected, as if there were, on the whole, an *attraction towards* the fixed body, or a *repulsion from it,* according as (§ 25) the translation is in the direction of the impulse or opposite to it. For, in each case, the impulse is altered by the introduction of an impulse *towards* the fixed body upon the moving body or bodies as they pass it; and (§ 23) the translation before and after the collision is always along the line of the impulse, and is altered in direction accordingly. This will be easily understood from the diagrams, where, in each case B represents the fixed body, the dotted line ITT′I′, and arrow-heads I I′, the directions of the force-resultant of the impulse at successive times, and the full arrow-heads T T′, the directions of the translation.

Fig. 1

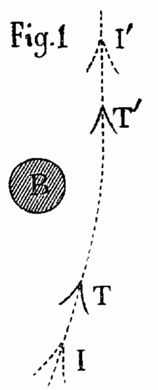
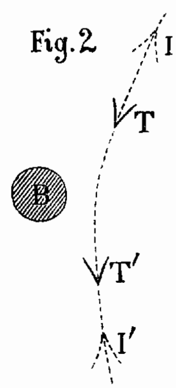



Fig. 2




All ordinary cases belong to the class illustrated by fig. 1. The case of a rigid ring, with cyclic motion (§ 25) established round it as core, belongs to the class illustrated by fig. 2, if the ring be projected through the fluid in the direction perpendicular to its own plane, and contrary to the cyclic motion through its centre.

34. When (§ 66) we substitute vortices for the moving solids, we shall see (§ 67) that the translation is probably always in the direction *with* the impulse. Hence, as illustrated by fig. 1, there is always the deflection, as if by *attraction,* when a group of vortices pass all on one side of a fixed body. This is easily observed, for a simple Helmholtz ring, by sending smoke rings on a large scale, according to

Professor TAIT's plan, in such directions as to pass very near a convex fixed surface. An ordinary 12-inch globe, taken off its bearings and hung by a thin cord, answers very well for the fixed body.

35. The investigation of §§ 30, 31, 32, is clearly applicable to a vortex or a moving body, or of a group of vortices or moving bodies, which keep always near one another (§ 23), passing near a projecting part of the fixed boundary, and being, before and after this collision (§ 29), at a very great distance from every part of the fixed boundary. Thus, a Helmholtz ring projected so as to pass near a projecting angle of two walls, shows a deflection of its course, as if caused by attraction towards the corner.

36. In every case the force-resultant of the impulse is, as we shall presently see (§ 37), determinate when the flow of the liquid across every element of any surface completely enclosing the solids or vortices is given; but not so, from such data, either the axis (§ 6) or the rotational moment, as we see at once by considering the case of a solid sphere (which may afterwards be supposed liquefied) set in motion by a force in any line not through the centre, and a couple in a plane perpendicular to it. For this line will be the "axis," and the impulsive couple will be the rotational moment of the whole motion of the solid and liquid. But the liquid, on all sides, will move exactly as it would if the impulse were merely an impulsive force of equal amount in a parallel line through the centre of the sphere, with therefore this second line for "axis" and zero for rotational moment. For illustration of rotational moment remaining latent in a liquid (with or without solids) until made manifest by actions, tending to alter its axis, or showing effects of centrifugal force due to it; see § 66, and others later.

37. The component impulse in any direction is equal to the corresponding component momentum of the mass enclosed within the surface S, containing all the places of application of the impulse, together with that of the impulsive pressure outwards on this surface. But as the matter enclosed by S (whether all liquid or partly liquid and partly solid) is of uniform density, its momentum will be equal to its mass multiplied into the velocity of the centre of gravity of the space within the surface S supposed to vary so as to enclose always the same matter, and will therefore depend solely on the normal motion of S; that is to say, on the component of the fluid velocity in the direction of the normal at every point of S. And the impulsive fluid pressure, corresponding to the generation of the actual motion from rest, being the time integral of the pressure during the instantaneous generation of the motion, is (§§ 31, 32) equal to $-\phi$, the velocity potential; which (§ 61) is determinate for every point of S, and of the exterior space when the normal component of the fluid motion is given for every point of S. Hence the proposition asserted in § 36. Denoting by $d\sigma$ any element of S; N the normal component of the fluid velocity; $\alpha$ the inclination to OX, of the normal drawn *outwards* through $d\sigma$; and X the $x$-component of the impulse; we

have for the two parts of this quantity considered above, and its whole value, the following expressions; of which the first is taken in anticipation from § 42—

$$\left.\begin{array}{ll} x\text{-momentum of matter, within S,} & =\iint \mathrm{N}x\,d\sigma \quad (8) \text{ of } \S\ 42 \\ x\text{-component of impulsive pressure on S, outwards,} & = -\iint \varphi \cos\alpha\, d\sigma \end{array}\right\} \quad (1).$$

$$\mathrm{X} = \iint (\mathrm{N}x - \varphi\cos\alpha)\, d\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

It is worthy of remark that this expression holds for the impulse of all the solids or vortices within S, even if there be others in the immediate neighbourhood outside: and that therefore its value must be zero if there be no solids or vortices within S, and N and $\phi$ are due solely to those outside.

38. If $\phi$ be the potential of a magnet or group of magnets, some within S and others outside it, and N the normal component magnetic force, at any point of S, the preceding expression (2) is equal to the $x$-component of the magnetic moment of all the magnets within S, multiplied by $4\pi$. For let $\rho$ be the density of any continuous distribution of positive and negative matter, having for potential, and normal component force, $\phi$ and N respectively, at every point of S. We have [Thomson & Tait, § 491 $(c)$] $\varrho = -\frac{1}{4\pi}\nabla^2\varphi$, and therefore

$$\iiint \varrho x\, dx\, dy\, dz = -\frac{1}{4\pi}\iiint x\left(\frac{d^2\varphi}{dx^2} + \frac{d^2\varphi}{dy^2} + \frac{d^2\varphi}{dz^2}\right) dx\, dy\, dz \quad . \quad . \quad (3).$$

Now, integrating by parts,* as usual with such expressions, we have

$$\iiint x\frac{d^2\varphi}{dx^2}\, dx\, dy\, dz = \iint x\frac{d\varphi}{dx}\, dy\, dz - \iiint \frac{d\varphi}{dx}\, dx\, dy\, dz = \iint \left(x\frac{d\varphi}{dx} - \varphi\right) dy\, dz\,.$$

Hence, integrating each of the other two terms of (3) once simply, and reducing as usual [Thomson & Tait, App. A $(a)$] to a surface integral, we have

$$\iiint \varrho x\, dx\, dy\, dz = -\frac{1}{4\pi}\iint (\mathrm{N}x - \varphi\cos\alpha)\, d\sigma \quad . \quad . \quad . \quad . \quad (4);$$

which proves the proposition, and also, of course, that if there be no matter within S, the value of the second member is zero.

39. Hence, considering the magnetic and hydrokinetic analogous systems with the sole condition that at every point of some particular closed surface, the magnetic potential is equal to the velocity potential, we conclude that $4\pi$ times the magnetic moment of all the magnetism within any surface, in the magnetic system, is equal to the force-resultant of the impulse of the solids or vortices within the corresponding surface in the hydrokinetic system; and that the directions of the magnetic axis and of the force-resultant of the impulse are the same. For the theory of magnetism, it is interesting to remark that indeterminate distributions of magnetism within the solids, or portions of fluid to which initiating

* The process here described leads merely to the equation obtained by taking the last two equal members of App. A (1) (Thomson & Tait) for the case $\alpha = 1$, $\mathrm{U} = \varphi$, $\mathrm{U}' = x$.

forces (§ 3) were applied, or determinate distributions in infinitely thin layers at their surfaces, may be found, which through all the space external to them shall produce the same potential as the velocity-potential, and therefore the same distribution of force as the distribution of velocity through the whole fluid. But inasmuch as when the magnetic force in the interior of a magnet is defined in the manner explained in § 48 (2) of my "Mathematical Theory of Magnetism,"* it is expressible through all space by the differential coefficients of a potential; and, on the contrary, for the kinetic system $u\,dx + v\,dy + w\,dz$ is not a complete differential generally through the spaces occupied by the solids, the agreement between resultant force and resultant flow holds only through the space exterior to the magnets and solids in the magnetic and kinetic systems respectively. But if the other definition of resultant force within a magnet, ["Math. Theory of Magnetism," § 77, foot-note, and § 78], published in preparation for a 6th chapter "On Electro-magnets" (still in my hands in manuscript, not quite completed), and which alone can be adopted for spaces occupied by non-magnetic matter traversed by electric currents, the magnetic force has not a potential within such spaces; and we shall see (§ 68) that determinate distributions of closed electric currents through spaces corresponding to the solids of the hydrokinetic system can be found which shall give for every point of space, whether traversed by electric currents or not, a resultant magnetic force, agreeing in magnitude and direction with the velocity, whether of solid or fluid, at the corresponding point of the hydrokinetic system. This thorough agreement for all space renders the electro-magnetic analogue preferable to the magnetic; and, having begun with the magnetic analogous system only because of its convenience for the demonstration of § 38, we shall henceforth chiefly use the purely electromagnetic analogue.

40. To prove the formula used in anticipation, in § 37 (1) we must now (§§ 41, 42, 43) find the momentum of the whole matter—fluid, fluid and solid, or even solid alone—at any instant within a closed surface S, in terms of the normal component velocity of the matter at any point of this surface, or, which is the same, the normal velocity of this surface itself, if we suppose it to vary so as to enclose always the same matter.

41. Let V be the volume of the space bounded by any varying closed surface S. As yet we need not suppose V constant. Let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the co-ordinates of of the centre of gravity. We have

$$V\bar{x} = \tfrac{1}{2}\iint[x^2\,dy\,dz] \quad . \quad . \quad . \quad . \quad (5),$$

where [ ] indicates that the expression within it is to be taken between proper limits for S. Now as S varies with the time, the area through which $\iint dy\,dz$ is taken will in general vary; but the increments or decrements which it experiences

* Trans. R. S. Lond., 1851; or "THOMSON's Electrical Papers." Macmillan. 1869.

at different parts of the boundary of this area, in the infinitely small time $dt$, contribute no increments or decrements to $\iint [x^2 dy\, dz]$, as we see most easily by first supposing S to be a surface everywhere convex outwards. Hence

$$\frac{d}{dt}\iint [x^2\, dy\, dz] = \iint \left[\frac{d\,(x^2)}{dt}\, dy\, dz\right] = 2\iint [x\,\frac{dx}{dt}\, dy\, dz] \quad . \quad (6).$$

But if N denote the velocity with which the surface moves in the direction of its outward normal at $(x, y, z)$, we have, in the preceding expression

$$\frac{dx}{dt} = \mathrm{N} \sec \alpha \quad . \quad . \quad . \quad . \quad (7),$$

if $\alpha$ be the inclination of the outward normal to OX. Hence

$$\frac{d(\mathrm{V}\bar{x})}{dt} = \iint [x\mathrm{N} \sec \alpha\, dy\, dz].$$

But the condition as to limits indicated by [ ] are clearly satisfied, if, $d\sigma$ denoting an element of the surface, such that

$$dy\, dz = \cos \alpha\, d\sigma,$$

we simply take $\iint d\sigma$ over the whole surface. Thus we have

$$\frac{d(\mathrm{V}\bar{x})}{dt} = \iint x\mathrm{N}\, d\sigma \quad . \quad . \quad . \quad . \quad (7);$$

42. In any case in which V is constant, this becomes

$$\mathrm{V}\,\frac{d\bar{x}}{dt} = \iint x\,\mathrm{N}\, d\sigma \quad . \quad . \quad . \quad . \quad (8).$$

If now the varying surface, S, is the boundary of a portion of the matter—fluid or solid—of uniform density unity, with whose motions we are occupied, the $x$-component momentum of this portion is $\mathrm{V}\frac{d\bar{x}}{dt}$; and, therefore, equation (8) is the required (§ 40) expression.

43. The same formulæ (7) and (8) are proved more shortly of course by the regular analytical process given by POISSON* and GREEN† in dealing with such subjects; thus, in short. Let $u, v, w$ be the components of velocity, of any matter, compressible or incompressible, at any point $(x, y, z)$ within S; and let $c$ denote the value at this point of $\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$, so that

$$\frac{du}{dx} = c - \left(\frac{dv}{dy} + \frac{dw}{dz}\right) \quad . \quad . \quad . \quad . \quad (9).$$

We have, for the component momentum of the whole matter within S, if of unit density at the instant considered,

$$\iiint u\, dx\, dy\, dz = \iint ux\, dy\, dz - \iiint x\,\frac{du}{dx}\, dx\, dy\, dz \quad . \quad (10).$$

* Théorie de la Chaleur, § 60.

† Essay on Electricity and Magnetism.

But by (9)

$$\iiint x \frac{du}{dx} dx\,dy\,dz = \iiint cx\,dx\,dy\,dz - \iiint x \left(\frac{dv}{dy} + \frac{dw}{dz}\right) dx\,dy\,dz$$

and by simple integrations,

$$\iiint x \left(\frac{dv}{dy} + \frac{dw}{dz}\right) dx\,dy\,dz = \iint x (v\,dx\,dz + w\,dx\,dy).$$

Using these in (10), and altering the expression to a surface integral, as in THOMSON & TAIT, App. A ($a$), we have

$$\begin{aligned}\iiint u\,dx\,dy\,dz &= \iint x\,(u\,dy\,dz + v\,dz\,dx + w\,dx\,dy) - \iiint cx\,dx\,dy\,dz \\ &= \iint x\,\mathrm{N}\,d\sigma - \iiint cx\,dx\,dy\,dz \quad . \quad . \quad . \quad . \quad . \quad (11),\end{aligned}$$

which clearly agrees with (7).

When this mass is incompressible, we have $c = o$ by the formula so ill named the equation "of continuity" (THOMSON & TAIT, § 191), and we fall upon (8.)

The proper analytical interpretation of the differential coefficients $\frac{du}{dx}$, &c., and of the equation of continuity, when, as at the surfaces of separation of fluid and solids, $u$, $v$, $w$ are discontinuous functions, having abruptly varying values, presents no difficulty.

44. In the theory of the impulse applied to the collision (§ 29) of solids or vortices moving through a liquid, the force-resultant of the impulse corresponds, as we have seen, precisely to the resultant momentum of a solid in the ordinary theory of impact. Some difficulty may be felt in understanding how the zero-momentum (§ 4) of the whole mass is composed; there being clearly positive momentum of solids and fluids in the direction of the impulse in some localities near the place of its application, and negative in others. [Consider, for example, the simple case of a solid of revolution struck by a single impulse in the line of its axis. The fluid moves in the direction of the impulse, before and behind the body, but in the contrary direction in the space round its middle.] Three modes of dividing the whole moving mass present themselves as illustrative of the distribution of momentum through it; and the following propositions (§ 45) with reference to them are readily proved (§§ 46, 47, 48).

45. I. Imagine any cylinder of finite periphery, not necessarily circular, completely surrounding the vortices (or moving solids), and any other surrounding none, and consider the infinitely long prisms of variously moving matter at any instant surrounded by these two cylinders. The component momentum parallel to the length of the first is equal to the component of the impulse parallel to the same direction; and that of the second is zero.

II. Imagine any two finite spherical surfaces, one enclosing all the vortices or moving solids, and the other none. The resultant-momentum of the whole matter enclosed by the first is in the direction of the impulse, and is equal to $\frac{2}{3}$ of its value. The resultant-momentum of the whole fluid enclosed by the second is the same as if it all moved with the same velocity, and in the same direction, as at its centre.

III. Imagine any two infinite planes at a finite distance from one another and from the field of motion, but neither cutting any solid or vortex. The component perpendicular to them of the momentum of the matter occupying at any instant the space between them (whether this includes none, some, or all of the vortices or moving solids) is zero.

46. To prove these propositions:—

I. Consider in either case a finite length of the prism extending to a very great distance in each direction from the field of motion, and terminated by plane or curved ends. Then, the motion being, as we may suppose (§ 61) started from rest by impulsive pressures on the solids [or (§ 66) on the portions of fluid constituting the vortices]; the impulsive fluid pressure on the cylindrical surface can generate no momentum parallel to the length; and to generate momentum in this direction there will be, in case 1, the impressed impulsive forces on the solids, and the impulsive fluid pressures on the ends; but in case 2 there will be only the impulsive fluid pressure on the ends. Now, the impulsive fluid pressures on the ends diminish [§ 50 (15)] according to the inverse square of the distance from the field of motion, when the prism is prolonged in each direction, and are therefore infinitely small when the prisms are infinitely long each way. Whence the proposition I.

47. By using the harmonic expansions § 19, (14), (15), in the several expressions (1), (2), of § 37, (1), (2); and the fundamental theorem

$$\iint \mathfrak{S}_i \mathfrak{S}_{i'} \, d\sigma = 0 ,$$

of the harmonic analysis [Thomson & Tait, App. B. (16)]; and putting $S_i = 0$ for one case, and $T_i = 0$ for the other; we prove the two parts of Prop. II., § 45 immediately.

48. To prove Prop. II., § 45, the well-known theory of electric images in a plane conductor* may be conveniently referred to. It shows that if $N_1$ denotes the normal component force at any point of an infinite plane due to any distribution, $\mu$, of matter in the space lying on one side of the plane, a distribution of matter over the plane having $\frac{1}{2\pi} N_1$ for surface density at each point exerts the same force as $\mu$ through all the space on the other side of the plane, and therefore that the whole quantity of matter in that surface distribution is equal to the

* Thomson, Camb. and Dub. Math. Journal, 1849; Liouville's Journal, 1845 and 1847; or Reprints of Electrical Papers, (Macmillan, 1869.)

whole quantity of matter in $\mu$.* Hence, $\iint d\sigma$, denoting integration over the infinite *plane*

$$\iint N_1 \, d\sigma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

if the whole quantity of matter in $\mu$ be zero. Hence, if N be the normal force due to matter through space on both sides of the plane, provided the whole quantity of matter on each side separately is zero,

$$\iint N \, d\sigma = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14);$$

since N is the sum of two parts, for each of which separately (12) holds. This translated into hydrokinetics, shows that the whole flow of matter across any infinite plane is zero at every instant when it cuts no solids or vortices. Hence, and from the uniformity of density which (§ 3), we assume, the centre of gravity of the matter between any two infinite fixed parallel planes, has no motion in the direction perpendicular to them at any time when no vortex or moving solid is cut by either: which is Prop. III. of § 4 in other words.

49. The integral flow of matter across any surface whatever, imagined to divide the whole volume of the finite fixed containing vessel of § 1 into two parts is necessarily zero, because of the uniformity of density; and therefore the momentum of all the matter bounded by two parallel planes, extending to the inner surface of the containing vessel, and the portion of this surface intercepted between them has always zero for its component perpendicular to these planes, whether or not moving solids or vortices are cut by either or both these planes. But it is remarkable that when any moving solid or vortex is cut by a plane, the integral flow of matter across this plane (if the containing vessel is infinitely distant on all sides from the field of motion), converges to a generally *finite* value, as the plane is extended to very great distances all round from the field of motion, which are still infinitely small in comparison with the distances to the containing vessel; and diminishes from that finite value to zero by another convergence, when the distances to which the plane is extended all round begin to be comparable with, and ultimately become equal to, the distances of the curve in which it cuts the containing vessel. Hence we see how it is that the condition of neither plane cutting any moving solid or vortex is necessary to allow § 46, III. to be stated without reference to the containing vessel, and are reminded that

* This is verified synthetically with ease, by direct integrations showing (whether by Cartesian or polar plane co-ordinates), that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{a\,dy\,dz}{(a^2+y^2+z^2)^{\frac{3}{2}}} = 2\pi \quad . \quad . \quad . \quad . \quad . \quad (13).$$

And taking $\frac{d}{da}$ of this, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(y^2+z^2-2a^2)dy\,dz}{(a^2+y^2+z^2)^{\frac{5}{2}}} = 0 \quad . \quad . \quad . \quad . \quad . \quad (12)'.$$

the synthesis of (12).

the equality to zero asserted in this proposition is proved in § 48 to be approximated to when the planes are extended to distances all round, which, though infinitely short of the distances to the containing vessel, are very great in comparison with their perpendicular distances from the most distant parts of the field of motion.

50. The convergencies concerned in § 46, I., III. may be analysed thus. Perpendicular to the resultant impulse draw any two planes on the two sides of the field of motion, with all the moving solids and vortices between them, and divide a portion of the space between them into finite prismatic portions by cylindrical (or plane) surfaces perpendicular to them. Suppose now one of these prismatic portions to include all the moving solids and vortices, and without altering the prismatic boundary, let the parallel planes be removed in opposite directions to distances each infinite (or very great) in comparison with distance of the most distant of the moving solids or vortices. By § 46, I., the momentum of the motion within this prismatic space is (approximately) equal to the force-resultant, I, of the impulse, and that of the motion within any one of the others is (approximately) zero.

But the sum of these (approximately) zero values must, on account of § 46, III., be equal to $-$I, if the portions of the planes containing the ends of the prismatic spaces be extended to distances very great in comparison with the distance between the planes. To understand this, we have only to remark that if $\phi$ denotes the velocity potential at a point distant D from the middle of the field, and $x$ from a plane through the middle perpendicular to the impulse, we have (§ 53) approximately,

$$\varphi = -\frac{\text{I}\,x}{4\pi\,\text{D}^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

provided D be great in comparison with the radius of the smallest sphere enclosing all the moving solids or vortices. Hence, putting $x = \pm a$ for the two planes under consideration, denoting by A the area of either end of one of the prismatic portions, and calling D *the proper mean distance* for this area, we have (§ 46) for the momentum of the fluid motion within this prismatic space, provided it contains no moving solids or vortices,

$$-2\,\frac{\text{I}\,a}{4\pi\,\text{D}^3}\,\text{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

This vanishes when $\frac{\text{A}}{\text{D}^2}$ is an infinitely small fraction (as $\frac{a}{\text{D}}$ is at most unity); but it is finite if $\frac{\text{A}}{\text{D}^2}$ is finite, provided $\frac{a}{\text{D}}$ be not infinitely small. And its integral value (compare § 48, footnote) converges to $-$I, when the portion of area included in the integration is extended till $\frac{a}{\text{D}}$ is infinitely small for all points of its boundary.

51. Both as regards the mathematical theory of the convergence of definite integrals, and as illustrating the distribution of momentum in a fluid, it is interesting to remark that, $u$ denoting component velocity parallel to $x$, at any point $(x, y, z)$, the integral $\iiint u\, dx\, dy\, dz$, expressing momentum, may, as is readily proved, have any value from $-\infty$ to $+\infty$ according to the portions of space through which it is taken.

52. As a last illustration of the distribution of momentum, let the containing vessel be spherical of finite radius $a$.

We have, as in § 19,

$$\left.\begin{aligned}\varphi = S_0 + S_1 r + S_2 r^2 + \&c., \\ + T_1 r^{-2} + T_2 r^{-3} + \&c.,\end{aligned}\right\} \quad \ldots \quad (14),$$

each series converging, provided $r$ is less than $a$, and greater than the radius of the smallest concentric spherical surface enclosing all the solids or vortices. Now, by the condition that there be no flow across the fixed containing surface, we must have

$$\frac{d\varphi}{dr} = 0, \text{ when } r = a \quad \ldots \quad (15),$$

which gives

$$S_i = \frac{i+1}{i}\frac{T_i}{a^{2i+1}} \quad \ldots \quad (16);$$

and (14) becomes

$$\varphi = \frac{T_1}{r^2}\left(1 + 2\frac{r^3}{a^3}\right) + \frac{T_2}{r^3}\left(1 + \frac{3}{2}\frac{r^5}{a^5}\right) + \&c. \quad \ldots \quad (17).$$

But [§ 37 (1) ] if the whole amount of the $x$-component of impulsive pressure exerted by the fluid within the spherical surface of radius $r$, upon the fluid round it be denoted by F, we have

$$F = -\iint \varphi \cos\theta\, d\sigma \quad \ldots \quad (18),$$

$\theta$ being the inclination to OX of the radius through $d\sigma$. Now $\cos\theta$ is a surface harmonic of the first order, and therefore all the terms of the harmonic expansion, except the first, disappear in the integral, which consequently becomes

$$F = -\left(1 + 2\frac{r^3}{a^3}\right)\iint T_1 \cos\theta \frac{d\sigma}{r^2} \quad \ldots \quad (19).$$

Now let

$$T_1 = -\frac{Ax + By + Cz}{r} \quad \ldots \quad (20),$$

this being [Thomson & Tait, App. B, §§ i, j] the most general expression for a surface harmonic of the first order. We have $\cos\theta = \frac{a}{r}$; and therefore (by spherical harmonics, or by the elementary analysis of moments of inertia of a uniform spherical surface),

$$-\iint T_1 \cos\theta \frac{d\sigma}{r^2} = \frac{A}{r^4}\iint x^2\, d\sigma = \frac{4\pi A}{3} \quad . \quad . \quad . \quad (21);$$

and (19) becomes

$$F = \left(1 + 2\frac{r^3}{a^3}\right) . \frac{4\pi A}{3} \quad . \quad . \quad . \quad . \quad (22):$$

Whence, if X denote the $x$-momentum of the fluid at any instant in the space between concentric spherical surfaces of radius $r$ and $r'$,

$$X = -\frac{2}{3}\frac{r^3 - r'^3}{a^3} 4\pi A \quad . \quad . \quad . \quad . \quad (23).$$

If $r$ and $r'$ be each infinitely small in comparison with $a$, this expression vanishes, as it ought to do, in accordance with § 45, II. But if

$$\left.\begin{array}{l} \frac{r}{a} = 0\,, \ \ \& \ r = a\,, \\ X = -\frac{2}{3} . 4\pi A \end{array}\right\} \quad . \quad . \quad . \quad . \quad (24),$$

it becomes

fulfilling § 4, by showing in the fluid outside the spherical surface of radius $r'$ a momentum equal and opposite to that (§ 45, II.) of the whole matter, whether fluid or solid, within that surface.

53. Comparing § 47 and § 52, we see that if X, Y, Z be rectangular components of the force-resultant of the impulse, the term $T_1 r^{-2}$ of the harmonic expansion (14) is as follows:—

$$T_1 r^{-2} = \frac{Xx + Yy + Zz}{4\pi r^3} \quad . \quad . \quad . \quad (25),$$

provided all the solids and vortices taken into account are within a spherical surface whose radius is very small in comparison with the distances of all other vortices or moving solids, and with the shortest distance to the fixed bounding surface.

54. HELMHOLTZ, in his splendid paper on Vortex Motion, has made the very important remark, that a certain fundamental theorem of GREEN'S, which has been used to demonstrate the determinateness of solutions in hydrokinetics, is subject to exception when the functions involved have multiple values. This calls for a serious correction and extension of elementary hydrokinetic theory, to which I now proceed.

55. In the general theorem (1) of THOMSON & TAIT, App. A let $a = 1$. It becomes

$$\iiint \left(\frac{d\varphi}{dx}\frac{d\varphi'}{dx} + \frac{d\varphi}{dy}\frac{d\varphi'}{dy} + \frac{d\varphi}{dz}\frac{d\varphi'}{dz}\right) dx\,dy\,dz = \iint d\sigma\, \varphi\, \mathfrak{d}\varphi' - \iiint dx\, dy\, dz\, \varphi \nabla^2 \varphi'$$

$$= \iint d\sigma \varphi' \mathfrak{d}\varphi - \iiint dx\, dy\, dz\, \varphi' \nabla^2 \varphi \quad . \quad . \quad . \quad (1),$$

which is true without exception if $\phi$ and $\phi'$ denote any two *single-valued* functions of $x, y, z$; $\iiint dx\, dy\, dz$ integration through the space enclosed by any finite closed surface, S; $\iint d\sigma$ integration over the area of this surface; and $\mathfrak{d}$ rate of variation per unit of length in the normal direction at any point of it. This is GREEN's original theorem, with HELMHOLTZ's limitation added (in italics.) The reader may verify it for himself.

56. But if either $\phi$ or $\phi'$ is a many-valued function, and the differential coefficients $\frac{d\phi}{dx}, \ldots, \frac{d\phi'}{dx}, \ldots,$ each single-valued, the double equation (1) cannot be generally true. Its first member is essentially unambiguous; but the process of integration by which the second member or the third member is found, would introduce ambiguity if $\phi$ or if $\phi'$ is many-valued. In one case the first member, though not equal to the ambiguous second, would be equal to the third, provided $\phi'$ is not also many-valued; and in the other, the first member, though not equal to the third, would be equal to the second, provided $\phi$ is not many-valued.

For example, let

$$\phi' = \tan^{-1}\frac{y}{x} \quad . \quad . \quad . \quad . \quad (2).$$

and let S consist of the portions of two planes perpendicular to OZ, intercepted between two circular cylinders having OZ for axis, and the portions of these cylinders intercepted between the two planes. The inner cylindrical boundary excludes from the space bounded by S, the line OZ where $\phi'$ has an infinite number of values, and $\frac{d\phi'}{dx}$, and $\frac{d\phi'}{dz}$ have infinite values. We have

$$\frac{d\phi'}{dx} = \frac{-y}{x^2+y^2},\ \frac{d\phi'}{dy} = \frac{x}{x^2+y^2} \quad . \quad . \quad . \quad (3),$$

and at every point of S, $d\phi' = 0$. Then, if $\phi$ be single-valued, there is no failure in the process proving the equality between the first and second members of (1), which becomes

$$\iiint \frac{x\frac{d\phi}{dy} - y\frac{d\phi}{dx}}{x^2+y^2}\, dx\, dy\, dz = 0 \quad . \quad . \quad . \quad (4).$$

Compare § 14 (6) to end.

The third member of (1) becomes

$$\iint d\sigma \tan^{-1}\frac{y}{x}\,\mathfrak{d}\phi - \iiint \tan^{-1}\frac{y}{x}\,\nabla^2\phi\, dx\, dy\, dz \quad . \quad . \quad (5),$$

which is no result of unambiguous integration of the first member through the space enclosed by S, as we see by examining, in this case, the particular meaning of each step of the ordinary process in rectangular co-ordinates for proving GREEN's theorem. It is thus seen that we must add to (5) a term

$$2\pi \iint dx\, dz \left(\frac{d\phi}{dy}\right)_{y=0},$$

if in its other terms the value of $\tan^{-1}\frac{y}{x}$ is reckoned continuously round from one side of the plane ZOX to the other: or

$$-2\pi\iint dy\,dz\left(\frac{d\phi}{dx}\right)_{x=0},$$

if the continuity be from one side of ZOY to the other; to render it really equal to the first member of (1). Thus, taking for example the first form of the added term, we now have for the corrected double equation (1) for the case of $\phi' = \tan^{-1}\frac{y}{x}$, $\phi$ any single valued function, and S the surface, composed of the two co-axal cylinders and two parallel planes specified above:

$$\iiint \frac{x\frac{d\phi}{dz} - y\frac{d\phi}{dx}}{x^2+y^2}\,dx\,dy\,dz = 0 = 2\pi\iint dx\,dz\left(\frac{d\phi}{dy}\right)_{y=0} + \iint d\sigma \tan^{-1}\frac{y}{x}\mathfrak{d}\phi$$
$$-\iiint dx\,dy\,dz\tan^{-1}\frac{y}{x}\nabla^2\phi \qquad (6).$$

But if we annex to S any barrier stopping circulation round the inner cylindrical core, all ambiguity becomes impossible, and the double equation (1) holds. For instance, if the barrier be the portion of the plane ZOX, intercepted between the co-axal cylinders and parallel planes constituting the S of § 55, so that $\iint d\sigma$ must now include integration over each side of this rectangular area; (6) becomes simply the strict application of (1) to the case in question.

57. The difficulty of the exceptional interpretation of GREEN's theorem for the class of cases exemplified in §§ 55 and 56, depends on the fact that $\int F ds$ may have different values when reckoned along the lengths of different curves, drawn within the space bounded by S, from a point P to a point Q; $ds$ being an infinitesimal element of the curve, and F the rate of variation of $\phi$ per unit of length along it. Let PCQ, PC′Q be two curves for which the $\int F ds$ has different values; and let both lie wholly within S. If we draw any curve from P to Q; make it first coincide with PCQ, and then vary it gradually until it coincides with PC′Q; it must in some of its intermediate forms cut the bounding surface S: for we have

$$F ds = \frac{d\phi}{dx}dx + \frac{d\phi}{dy}dy + \frac{d\phi}{dz}dz$$

throughout the space contained within S, and $\frac{d\phi}{dx}, \frac{d\phi}{dy}, \frac{d\phi}{dz}$, are each of them unambiguous by hypothesis; which implies that $\int F ds$ has equal values for all gradual variations of one curve between P and Q, each lying wholly within S. Now, in a simply continuous space, a curve joining the points P and Q may be gradually varied from any curve PCQ to any other PC′Q, and therefore if the space contained within S be simply continuous, the difficulty depending on the multiplicity of value of $\phi$ or $\phi'$ cannot exist. And however multiply continuous (§ 58) the space may be, the difficulty may be evaded if we annex to S a surface or surfaces stopping every aperture or passage on the openness of which its multiple continuity depends; for these annexed surfaces, as each of them occupies no space, do not disturb the triple integrations (1), and will, therefore, not alter the values of its first member; but by removing the multiplicity of continuity, they free each of the integrations by parts, by which its second or third members are obtained, from all ambiguity. To avoid circumlocution, we shall call $\beta$ the addition thus made to S; and further, when the space within S is (§ 58) not merely doubly but triply, or quadruply, or more multiply, continuous, we shall designate by $\beta_1$, $\beta_2$; or $\beta_1$, $\beta_2$, $\beta_3$; and so on; the several parts of $\beta$ required in any case to stop all multiple continuity of the space. These parts of $\beta$ may be quite detached from one another, as when the multiple continuity is that due to detached rings, or separate single tunnels in a solid. But one part $\beta_1$ may cut through part of another, $\beta_2$, as when two rings (§ 58, diagram) linked into one another without touching constitute part of the boundary of the space considered. And we shall denote by $\iint d\varsigma$, integration over the surface $\beta$, or over any one of its parts, $\beta_1$, $\beta_2$, &c. Let now P and Q be each infinitely near a point B, of $\beta$, but on the two sides of this surface. Let $\kappa$ denote the value of $\int \mathrm{F}ds$ along any curve lying wholly in the space bounded by S, and joining PQ without cutting the barrier; this value being the same for all such curves, and for all positions of B to which it may be brought without leaving $\beta$, and without making either P or Q pass through any part of $\beta$. That is to say, $\kappa$ is a single constant when the space is not more than doubly continuous; but it denotes one or other of $n$ constants $\kappa_1, \kappa_2, \ldots \kappa_n$, which may be all different from one another, when the space is $n$-ply continuous. Lastly, let $\kappa'$ denote the same element, relatively to $\phi'$, as $\kappa$ relatively to $\phi$. We find that the first steps of the integrations by parts now introduce, without ambiguity, the additions

$$\Sigma\kappa \iint d\varsigma\, \mathfrak{d}\varphi', \text{ and } \Sigma\kappa' \iint d\varsigma\, \mathfrak{d}\varphi \quad . \quad . \quad . \quad (6),$$

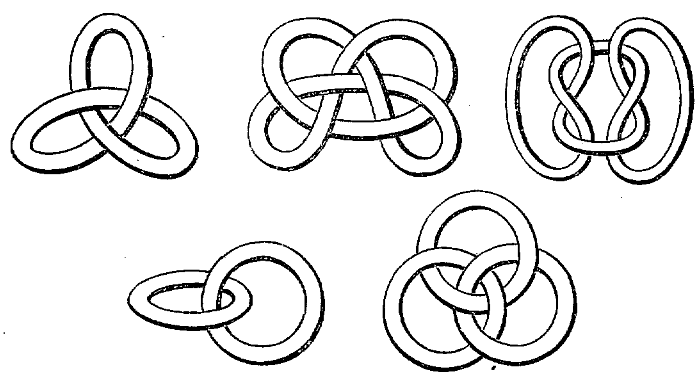
to the second and third numbers of (1): $\Sigma$ denoting summation of the integrations for the different constituents $\beta_1$, $\beta_2$, ... of $\beta$; but only a single term when the space is (§ 58) not more than doubly continuous. GREEN's theorem thus corrected becomes

$$\iint\!\!\Big(\frac{d\varphi}{dx}\frac{d\varphi'}{dx} + \frac{d\varphi}{dy}\frac{d\varphi'}{dy} + \frac{d\varphi}{dz}\frac{d\varphi'}{dz}\Big)\,dx\,dy\,dz = \iint d\sigma\, \varphi\mathfrak{d}\varphi' + \Sigma\kappa\iint d\varsigma\,\mathfrak{d}\varphi' - \iiint \varphi\,\nabla^2\varphi'\,dx\,dy\,dz$$

$$= \iint d\sigma\, \varphi'\mathfrak{d}\varphi + \Sigma\kappa'\iint d\varsigma\,\mathfrak{d}\varphi - \iiint \varphi'\,\nabla^2\varphi\,dx\,dy\,dz \quad . \quad (7).$$

58. Adopting the terminology of RIEMANN, as known to me through HELMHOLTZ, I shall call a finite position of space $n$-ply continuous when its bounding surface is such that there are $n$ irreconcilable paths between any two points in it. To prevent any misunderstanding, I add (1), that by *a* portion of space I mean such a portion that any point of it may be travelled to from any other point of it, without cutting the bounding surface; (2), that the "paths" spoken of all lie within the portion of space referred to; and (3), that by irreconcilable paths between two points P and Q; I mean paths such, that a line drawn first along one of them cannot be gradually changed till it coincides with the other, being always kept passing through P and Q, and always wholly within the portion of space considered. Thus, when all the paths between any two points are reconcilable, the space is simply continuous. When there are just two sets of paths, so that each of one set is irreconcilable with any one of the other set, the space is doubly continuous; when there are three such sets it is triply continuous, and so on. To avoid circumlocutions, we shall suppose S to be the boundary of a hollow space in the interior of a solid mass, so thick that no operations which we shall consider shall ever make an opening to the space outside it. A tunnel through this solid opening at each end into the interior space constitutes the whole space doubly continuous; and if more tunnels be made, every new one adds one to the degree of multiple continuity. When one such tunnel has been made, the surface of the tunnel is continuous with the whole bounding surface of the space considered; and in reckoning degrees of continuity, it is of no consequence whether the ends of any fresh tunnel be in one part or another of this whole surface. Thus, if two tunnels be made side by side, a hole anywhere opening from one of them into the other adds one to the degree of multiple continuity. Any solid detached from the outer bounding solid, and left, whether fixed or movable in the interior space, adds to the bounding surface an isolated portion, but does not interfere with the reckoning of multiple continuity. Thus, if we begin with a simply continuous space bounded outside by the inner surface of the supposed external solid, and internally by the boundary of the detached solid in its interior, and if we drill a hole in this solid we produce double continuity. Two holes, or two solids in the interior each with one hole (such as two ordinary solid rings), constitute triple continuity, and so on. A sponge-like solid whose pores communicate with one another, illustrates a high degree of multiple continuity, and it is of no consequence whether it is attached to the external bounding solid or is an isolated solid in the interior. Another type of multiple continuity, that presented by two rings linked in one another, was referred to in § 57.

When many rings are linked into one another in various combinations, there are complicated mutual intersections of the several partial barriers $\beta_1, \beta_2, \ldots$ required to stop all multiple continuity. But without having any portion of the bounding solid detached, as in that case in which one at least of the two rings is loose, we have varieties of multiple continuity curiously different from that illustrated by a single ordinary straight or bent tunnel, illustrated sufficiently by the simplest types, which are obtained by boring a tunnel along a line agreeing in form with the axis of a cord or wire on which a simple knot is tied; and by fixing the two ends of wire with a knot on it to the bounding solid, so that the surface of the wire shall become part of the bounding surface of the space considered, the knot not being pulled tight, and the wire being arranged not to touch itself in any point; or by placing a knotted wire, with its ends united, in the interior of the space. No amount of knotting or knitting, however complex, in the cord whose axis indicates the line of tunnel, complicates in any way the continuity of the space considered, or alters the simplicity of the barrier surface required to stop the circulation. But it is otherwise when a knotted or knitted wire forms part of the bounding solid. A single simple knot, though giving only double continuity, requires a curiously self-cutting surface for stopping barrier: which, in its form of minimum area, is beautifully shown by the liquid film adhering to an endless wire, like the first figure, dipped in a soap solution and removed. But no complication of these types, or of combinations of them with one another, eludes the statements and formulæ of § 57.



59. I shall now give a dynamical lemma, for the immediate object of preparing to apply Green's corrected theorem (§ 57) to the motion of a liquid through a multiply continuous space. But later we shall be led by it to very simple demonstrations of Helmholtz's fundamental theorems of vortex motion; and shall see that it may be used as a substitute for the common equations of hydrokinetics.

(Lemma). An endless finite tube* of infinitesimal normal section, being given full of liquid (whether circulating round through it, or at rest) is altered in shape,

* A finite length of tube with its ends done away by uniting them together.

*Instalment, received Nov.—Dec.* 1869 [§ 59 - § 64 (5)].

length, and normal section, in any way, and with any speed. The average value of the component velocity of the fluid along the tube, reckoned all round the circuit (irrespectively of the normal section), varies inversely as the length of the circuit.

59. (*a*). To prove this, consider first a single particle of unit mass, acted on by any force, and moving along a smooth guiding curve, which is moved and bent about quite arbitrarily. Let $\varrho$ be the radius of curvature, and $\xi, \eta$ the component velocities of the guiding curve, towards the centre of curvature, and perpendicular to the plane of curvature, at the point P, through which the moving particle is passing at any instant. Let $\zeta$ be the component velocity of the particle itself, along the instantaneous direction of the tangent through P. Thus $\xi, \eta, \zeta$ are three rectangular components of the velocity of the particle itself. Let $\mathfrak{Z}$ be the component in the direction of $\zeta$, of the whole force on P. We have, by elementary kinetics,

$$\frac{d\zeta}{dt} = \mathfrak{Z} + \frac{\zeta\xi}{\varrho} + \xi\frac{d\xi}{ds} + \eta\frac{d\eta}{ds} \qquad . \quad . \quad . \quad . \qquad (1),*$$

* This theorem (not hitherto published?) will be given in the second volume of THOMSON and TAIT's "Natural Philosophy." It may be proved analytically from the general equations of the motion of a particle along a varying guide-curve (WALTON, "Cambridge Mathematical Journal," 1842, February); or more synthetically, thus—Let $l, m, n$ be the direction cosines of PT, the tangent to the guide at the point through which the particle is passing at any instant; $(x, y, z)$ the co-ordinates of this point, and $(\dot{x}, \dot{y}, \dot{z})$ its component velocities parallel to fixed rectangular axes. We have

$$\zeta = l\dot{x} + m\dot{y} + n\dot{z}\,; \text{ and } \mathfrak{Z} = l\ddot{x} + m\ddot{y} + n\ddot{z}\,,$$

and from this

$$\frac{d\zeta}{dt} = l\ddot{x} + m\ddot{y} + n\ddot{z} + \dot{l}\dot{x} + \dot{m}\dot{y} + \dot{n}\dot{z} = \mathfrak{Z} + \dot{l}\dot{x} + \dot{m}\dot{y} + \dot{n}\dot{z}\,.$$

But it is readily proved (THOMSON and TAIT's "Natural Philosophy, § 9, to be made more explicit on this point in a second edition) that the angular velocity with which PT changes direction is equal to $\sqrt{(\dot{l}^2 + \dot{m}^2 + \dot{n}^2)}$, and, if this be denoted by $\omega$, that

$$\frac{\dot{l}}{\omega}, \ \frac{\dot{m}}{\omega}, \ \frac{\dot{n}}{\omega}$$

are the direction cosines of the line PK, perpendicular to PT in the plane in which PT changes direction, and on the side towards which it turns. Hence,

$$\frac{d\zeta}{dt} = \mathfrak{Z} + \kappa\omega$$

if $\kappa$ denote the component velocity of P along PK. Now, if the curve were fixed we should have $\omega = \frac{\zeta}{\varrho}$, by the kinematic definition of curvature (THOMSON and TAIT, § 5); and the plane in which PT changes direction would be the plane of curvature. But in the case actually supposed, there is also in this plane an additional angular velocity equal to $\frac{d\xi}{ds}$, and a component angular velocity in the plane of PT and $\eta$, equal to $\frac{d\eta}{ds}$; due to the normal motion of the varying curve. Hence the whole angular velocity $\omega$ is the resultant of two components,

$$\frac{\zeta}{\varrho} + \frac{d\xi}{ds} \text{ in the plane of } \xi\,,$$

where $\varrho$ denotes the radius of curvature, and $\frac{d\xi}{ds}$, $\frac{d\eta}{ds}$ rates of variation of $\xi$ and $\eta$ from point to point along the curve at one time.

59. (*b*). Now, instead of a single particle of unit mass, let an infinitesimal portion, $\mu$, of a liquid, filling the supposed endless tube, be considered. Let $\varpi$ be the area of the normal section of the tube in the place where $\mu$ is, and $\delta s$ the length along the tube of the space occupied by it, at any instant; so that (as the density of the fluid is called unity),

$$\mu = \varpi \delta s .$$

Further, let $\frac{dp}{ds}$ denote the rate of variation of the fluid pressure along the tube, so that

$$\mathfrak{Z} = -\varpi \frac{dp}{ds} \delta s .$$

Thus we have, by (1),

$$\frac{d\zeta}{dt} = \frac{\zeta\xi}{\varrho} + \xi \frac{d\xi}{ds} + \eta \frac{d\eta}{ds} - \frac{dp}{ds} \quad . \quad . \quad . \quad . \quad (2).$$

(*c*). Now, because the two ends of the arc $\delta s$ move with the fluid, we have, by the kinematics of a varying curve,

$$\frac{d\delta s}{dt} = \frac{d\zeta}{ds} \delta s - \frac{\xi}{\varrho} \delta s \quad . \quad . \quad . \quad . \quad . \quad . \quad (3);$$

and, therefore,

$$\frac{d(\zeta\delta s)}{dt} = \frac{d\zeta}{dt} \delta s + \zeta \left( \frac{d\zeta}{ds} \delta s - \frac{\xi}{\varrho} \delta s \right) \quad . \quad . \quad . \quad . \quad (4).$$

Substituting in this for $\frac{d\zeta}{dt}$ its value by (2), we have

$$\frac{d(\zeta\delta s)}{dt} = \left( \xi \frac{d\xi}{ds} + \eta \frac{d\eta}{ds} - \frac{dp}{ds} + \zeta \frac{d\zeta}{ds} \right) \delta s ,$$

or

$$\frac{d(\zeta\delta s)}{dt} = \delta(\tfrac{1}{2}q^2 - p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

if $q$ denote the resultant fluid velocity; and $\delta$, differences for the two ends of the arc $\delta s$. Integrating this through the length of any finite arc $P_1P_2$ of the fluid, its ends $P_1$, $P_2$, moving with the fluid, we have

$$\frac{d\Sigma_1^2(\zeta\delta s)}{dt} = (\tfrac{1}{2}q^2 - p)_2 - (\tfrac{1}{2}q^2 - p)_1 \quad . \quad . \quad . \quad . \quad . \quad (6),$$

the suffixes denoting the values of the bracketed function, at the points $P_2$ and

---

and

$$\frac{d\eta}{ds} \text{ in the plane of } \eta .$$

Hence

$$\xi \left( \frac{\zeta}{\varrho} + \frac{d\xi}{ds} \right) + \eta \frac{d\eta}{ds} = \kappa\omega ,$$

and the formula (1) of the text is proved.

$P_1$, respectively; and $\Sigma_1^2$ denoting integration along the arc from $P_1$ to $P_2$. Let now $P_2$ be moved forward, or $P_1$ backward, till these points coincide, and the arc $P_1P_2$ becomes the complete circuit; and let $\Sigma$ denote integration round the whole closed circuit. (6) becomes

$$\frac{d\Sigma(\zeta\delta s)}{dt} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7);$$

and we conclude that $\Sigma\zeta\delta s$ remains constant, however the tube be varied. This is the proposition to be proved, as the "average velocity" referred to is found by dividing $\Sigma(\zeta\delta s)$ by the length of the tube.

59. (*d*). The tube, imagined in the preceding, has had no other effect than exerting, by its inner surface, normal pressure on the contained ring of fluid. Hence the proposition* at the beginning of § 59 is applicable to any closed ring of fluid forming part of an incompressible fluid mass extending in all directions through any finite or infinite space, and moving in any possible way; and the formulæ (5) and (6) are applicable to any infinitesimal or infinite arc of it with two ends not met. Thus in words—

PROP. (1.) *The line-integral of the tangential component velocity round any closed curve of a moving fluid remains constant through all time.*

And, PROP. (2), The rate of augmentation, per unit of time, of the space integral of the velocity along any terminated arc of the fluid is equal to the

* Equation (6), from which, as we have seen, that proposition follows immediately, may be proved with greater ease, and not merely for an incompressible fluid, but for any fluid in which the density is a function of the pressure, by the method of rectilineal rectangular co-ordinates from the ordinary hydrokinetic equations. These equations are—

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{d\varpi}{dx}, \quad \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{d\varpi}{dy}, \quad \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{d\varpi}{dz},$$

if $\frac{\mathrm{D}}{\mathrm{D}t}$ denote rate of variation per unit of time, of any function depending on a point or points *moving with the fluid;* and $\varpi = \int \frac{dp}{\varrho}$, $\varrho$ denoting density. In terms of rectangular rectilineal co-ordinates we have

$$\zeta\delta s = u\delta x + v\delta y + w\delta z.$$

Hence

$$\frac{\mathrm{D}(\zeta\delta s)}{\mathrm{D}t} = \frac{\mathrm{D}u}{\mathrm{D}t}\delta x + u\frac{\mathrm{D}\delta x}{\mathrm{D}t} + \&c.$$

Now

$$\frac{\mathrm{D}\delta x}{\mathrm{D}t} = \delta u, \quad \frac{\mathrm{D}\delta y}{\mathrm{D}t} = \delta v, \quad \text{and } \frac{\mathrm{D}\delta z}{\mathrm{D}t} = \delta w.$$

These and the kinetic equations reduce the preceding to

$$\frac{\mathrm{D}(\zeta\delta s)}{\mathrm{D}t} = u\delta u + v\delta v + w\delta w - \frac{d\varpi}{dx}\delta x - \frac{d\varpi}{dy}\delta y - \frac{d\varpi}{dz}\delta z = \delta\left[\tfrac{1}{2}(u^2 + v^2 + w^2) - \varpi\right] \quad . \quad (8);$$

whence, by $\Sigma$ integration, equation (6) generalised to apply to compressible fluids.

excess of the value of $\frac{1}{2}q^2 - p$, at the end towards which tangential velocity is reckoned as positive, above its value at the other end.

59. (*e*). The condition that $u\,dx + v\,dy + w\,dz$ is a complete differential [proved above (§ 13) to be the criterion of irrotational motion] means simply

*That the flow* [defined § 60 (*a*)] *is the same in all different mutually reconcilable lines from one to another of any two points in the fluid;* or, which is the same thing,

*That the circulation* [§ 60 (*a*)] *is zero round every closed curve capable of being contracted to a point without passing out of a portion of the fluid through which the criterion holds.*

From Proposition (1), just proved, we see that this condition holds through all time for any portion of a moving fluid for which it holds at any instant; and thus we have another proof of LAGRANGE's celebrated theorem (§ 16), giving us a new view of its dynamical significance, which [see for example § 60 (*g*)] we shall find of much importance in the theory of vortex motion.

(*f*). But it is only in a closed curve, *capable of being contracted to a point without passing out of space occupied by irrotationally moving fluid,* that the circulation is necessarily *zero,* in irrotational motion. In § 57 we saw that a continuous fluid mass, occupying doubly or multiply continuous space, may move altogether irrotationally, yet so as to have finite circulation in a closed curve PP′QQ′P, provided PP′Q and PQ′Q are "irreconcilable paths" between P and Q. *That the circulation must be the same in all mutually reconcilable closed curves* (compare § 57), is an immediate consequence from the now proved [§ 59 (Prop. 2)] equality of the flows [§ 60 (*a*)] in all mutually reconcilable conterminous arcs. For by leaving one part of a closed curve unchanged, and varying the remaining arc continuously, no change is produced in the flow, in this part; and, by repetitions of the process, a closed curve may be changed to any other reconcilable with it.

60. *Definitions and elementary propositions* (*a*). The line-integral of the tangential component velocity along any finite line, straight or curved, in a moving fluid, is called the flow in that line. If the line is endless (that is, if it forms a closed curve or polygon), the *flow* is called *circulation.* The use of these terms abbreviates the statements of Propositions (2) and (1) of § 59 to the following:—

[§ 59, Prop. (2)]. The rate of augmentation, per unit of time, of the flow in any terminated line which moves with the fluid, is equal to the excess of the value of $\frac{1}{2}q^2 - p$ at the end from which, above its value at the end towards which, positive flow is reckoned.

[§ 59, Prop. (1)]. The circulation in any closed line moving with the fluid, remains constant through all time.

(*b*). If any open finite surface, lying altogether within a fluid, be cut into parts by lines drawn across it, the circulation in the boundary of the whole is equal to the sum of the circulations in the boundaries of the parts. This is obvious, as the latter sum consists of an equal positive and negative flow in each portion of boundary common to two parts, added to the sum of the flows in all the parts into which the single boundary of the whole is divided.

60. (*c*). Hence the circulation round the boundaries of infinitesimal areas, infinitely near one another in one plane, are simply proportional to these areas.

(*d*). *Proposition.* Let any part of the fluid rotate as a solid (that is, without changing shape); or consider simply the rotation of a solid. The "circulation" in the boundary of any plane figure moving with it is equal to twice the area enclosed, multiplied by the component angular velocity in that plane (or round an axis perpendicular to that plane). For, taking $r$, $\theta$ to denote polar co-ordinates of any point in the boundary, A the enclosed area, and $\omega$ the component angular velocity in the plane, and continuing the notation of § 59, we have

$$\zeta = r\omega \frac{rd\theta}{ds},$$

and therefore

$$\Sigma\zeta\delta s = \omega\Sigma r^2 \frac{d\theta}{ds}\delta s = \omega\Sigma r^2\delta\theta = \omega \times 2\text{A}.$$

(*e*). *Definition.* (For a fluid moving in any manner), the circulation round the boundary of an infinitesimal plane area, divided by double the area, is called the *component rotation* in that plane (or round an axis perpendicular to that plane) of the neighbouring fluid.

In this statement, the single word "rotation" is used for *angular velocity of rotation:* and the definition is justified by (*c*) and (*d*); also by § 13 (2) above, applied to (*p*) below. It agrees, in virtue of (*p*), with the definition of rotation in fluid motion given first of all, I believe, by STOKES, and used by HELMHOLTZ in his memorable "Vortex Motion," also in THOMSON and TAIT'S "Natural Philosophy," §§ 182 and 190 (*j*).

(*f*). *Proposition.* If $\xi$, $\eta$, $\zeta$ be the components of rotation at any point, P, of a fluid, round three axes at right angles to one another, and $\omega$ the component round an axis, making with them angles whose cosines are $l$, $m$, $n$,

$$\omega = \xi l + \eta m + \zeta n.$$

To prove this, let a plane perpendicular to the last-mentioned axis cut the other three in A, B, C. The circulation in the periphery of the triangle ABC is, by (*b*), equal to the sum of the circulations in the peripheries PBC, PCA, and PAB. Hence, calling $\Delta$ and $\alpha$, $\beta$, $\gamma$ the areas of these four triangles, we have, by (*e*),

$$\omega\Delta = \xi\alpha + \eta\beta + \zeta\gamma.$$

But $\alpha$, $\beta$, $\gamma$ are the projections of $\Delta$ on the planes of the pairs of the rectangular axes; and so the proposition is proved.

It follows, of course, that the composition of rotations in a fluid fulfils the law of the compositions of angular velocities of a solid, of linear velocities, of forces, &c.

60. (*g*). Hence, in any infinitesimal part of the fluid, the circulation is zero in the periphery of every plane area passing through a certain line;—the resultant axis of rotation of that part of the fluid. But (*a*) the circulation remains zero in every closed line moving with the fluid, for which it is zero at any time. Hence

(*h*). The axial lines [defined (*i*)] move with the fluid.

(*i*). *Definition.* An axial line through a fluid moving rotationally, is a line (straight or curved) whose direction at every point coincides with the resultant axis of rotation through that point.

(*j*). *Proposition.* The resultant rotation of any part of the fluid varies in simple proportion to the length of an infinitesimal arc of the axial line through it, terminated by points moving with the fluid. To prove this, consider any infinitesimal plane area, A, moving with the fluid. Let $\omega$ be the resultant rotation, and $\theta$ the angle between its axis and the perpendicular to the plane of A. This makes $\omega \cos \theta$ the component rotation in the plane of A; and therefore $A\omega \cos \theta$ remains constant. Now, draw axial lines through all points of the boundary of A, forming a tube whose area of normal section is $A \cos \theta$. The resultant rotation must vary inversely as this area, and therefore (in consequence of the incompressibility of the fluid) directly as the length of an infinitesimal line along the axis.

(*k*). Form a surface by axial lines drawn through all points of any curve in the fluid. The circulation is zero round the boundary of any infinitesimal area of this surface; and therefore (*b*) it is zero round the boundary of any finite area of it.

(*l*). Let the curve of (*k*) be closed, and therefore the surface tubular. On this surface let ABCA, A′B′C′A′ be any two curves closed round the tube, and ADA′ any arc from A to A′. The circulation in the closed path, ADA′B′C′A′DACBA, is zero by (*h*). Hence the circulation in ABCA is equal to the circulation in A′B′C′A′—that is to say,

The circulations are equal in all circuits of a vortex tube.

(*m*). *Definitions.* An *axial surface* is a surface made up of axial lines. A *vortex tube* is an axial surface through every point of which a finite endless path, cutting every axial line it meets, can be drawn. Any such path, passing just once round, is called *a circuit*, or *the circuit* of the tube. The *rotation of a vortex tube* is the circulation in its circuit. A *vortex sheet* is (a portion as it were of a collapsed vortex tube) a surface on the two sides of which the fluid moves with different tangential component velocities.

60. (*n.*) Draw any surface cutting a vortex tube, and bounded by it. The surface integral of the component rotation round the normal has the same value for all such surfaces; and this common value is what we now call the rotation of the tube.

(*o*). In an unbounded infinite fluid, an axial tube must be either finite and endless or infinitely long in each direction.* In an infinite fluid with a boundary (for instance, the surface of an enclosed solid), an axial tube may have two ends, each in the boundary surface; or it may have one end in the boundary surface, and no other; or it may be infinitely long in each direction, or it may be finite and endless. In a finite fluid mass, an axial tube may be endless, or may have one end, but, if so, must have another, both in the boundary surface.

(*p*). *Proposition.* Applying the notation of (*f*), to axes parallel to those of co-ordinates $x$, $y$, $z$, and denoting, as formerly, by $u$, $v$, $w$, the components of the fluid velocity at $(x, y, z)$, we have—

$$\xi = \tfrac{1}{2}\left(\frac{dw}{dy} - \frac{dv}{dz}\right), \quad \eta = \tfrac{1}{2}\left(\frac{du}{dz} - \frac{dw}{dx}\right), \quad \zeta = \tfrac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy}\right).$$

The proof is obvious, according to the plan of notation, &c., followed in § 13 above.

(*q*). Hence by (*f*), (*e*), and (*b*)—

$$\iint d\mathrm{S}\left\{ l\left(\frac{dw}{dy} - \frac{dv}{dz}\right) + m\left(\frac{du}{dz} - \frac{dw}{dx}\right) + n\left(\frac{dv}{dx} - \frac{du}{dy}\right)\right\} = \int (udx + vdy + wdz).$$

where $\iint d\mathrm{S}$ denotes integration over any portion of surface bounded by a closed curve; $\int(udx$ + &c.) integration round the whole of this curve; and $(l, m, n)$ the direction cosines of any point $(x, y, z)$ in the surface. It is worthy of remark that the equation of continuity for an incompressible fluid does not enter into the demonstration of this proposition, and therefore $u$, $v$, $w$ may be any functions whatever of $x$, $y$, $z$. In a purely analytical light, the result has an important bearing on the theory of the integration of complete or incomplete differentials. It was first given, with the indication of a more analytical proof than the preceding, in THOMSON and TAIT'S "Natural Philosophy," § 190 (*j*).

(*r*). Propositions (*h*) (*j*) (*n*) (*o*) of the present section (§ 60) are due to HELMHOLTZ; and with his integration for associated rotational and cyclic irrotational motion in an unbounded fluid, to be given below, constitute his general theory of vortex motion. (*n*) and (*o*) are purely kinematical; (*h*) and (*j*) are dynamical.

(*s*). Henceforth I shall call *a circuit* any closed curve not continuously reducible to a point, in a multiply continuous space. I shall call *different circuits*, any

* Vortex tubes apparently ending in the fluid, for instance, a portion of fluid bounded by a figure of revolution, revolving round its axis as a solid, constitute no exception. Each infinitesimal vortex tube in this case is completed by a strip of vortex sheet and so is endless.

two such closed curves if mutually irreconcilable (§ 58); but different mutually reconcilable closed curves will not be called different circuits.

60. (*t*). Thus, $(n+1)$ply continuous space, is a space for which there are $n$, and only $n$, different circuits. This is merely the definition of § 58, abbreviated by the definite use of the word circuit, which I now propose. The general terminology regarding simply and multiply continuous spaces is, as I have found since § 58 was written, altogether due to HELMHOLTZ; RIEMANN's suggestion, to which he refers, having been confined to two-dimensional space. I have deviated somewhat from the form of definition originally given by HELMHOLTZ, involving, as it does, the difficult conception of a stopping barrier;* and substituted for it the definition by reconcilable and irreconcilable paths. It is not easy to conceive the stopping barrier of any one of the first three diagrams of § 58, or to understand its singleness; but it is easy to see that in each of those three cases, any two closed curves drawn round the solid wire represented in the diagrams are reconcilable, according to the definition of this term given in § 58, and therefore, that the presence of any such solid adds only one to the degree of continuity of the space in which it is placed.

(*u*). If we call a *partition*, a surface which separates a closed space into two parts, and, as hitherto, a *barrier*, any surface edged by the boundary of the space, HELMHOLTZ's definition of multiple continuity may be stated shortly thus:—

*A space is* (n+1)*ply continuous if* n *barriers can be drawn across it, none of which is a partition.*

(*v*). HELMHOLTZ has pointed out the importance in hydrokinetics of many-valued functions, such as $\tan^{-1}\frac{y}{x}$, which have no place in the theories of gravitation, electricity, or magnetism, but are required to express electro-magnetic potentials, and the velocity potentials for the part of the fluid which moves irrotationally in vortex motion. It is, therefore, convenient, before going farther, that we should fix upon a terminology, with reference to functions of that kind, which may save us circumlocutions hereafter.

(*w*). A function $\phi(x, y, z)$ will be called *cyclic* if it experiences a constant augmentation every time a point P, of which $x$, $y$, $z$ are rectangular rectilineal co-ordinates, is carried from any position round a certain circuit to the same position again, without passing through any position for which either $\frac{d\phi}{dx}$, $\frac{d\phi}{dy}$, or $\frac{d\phi}{dz}$ becomes infinite. The value of this augmentation will be called the cyclic

* But without this conception we can make no use of the theory of multiple continuity in hydrokinetics (see §§ 61–63), and HELMHOLTZ's definition is, therefore, perhaps preferable after all to that which I have substituted for it. Mr CLERK MAXWELL tells me that J. B. LISTING has more recently treated the subject of multiple continuity in a very complete manner in an article entitled "Der Census räumlicher Complexe."—*Königl. Ges. Göttingen*, 1861. See also Prof. CAYLEY "On the Partition of a Close."—*Phil. Mag.* 1861.

constant for that particular circuit. The cyclic constant must clearly have the same value for all circuits mutually reconcilable (§ 58), in space throughout which the three differential coefficients remain all finite.

60. ($x$). When the function is cyclic with reference to several different mutually irreconcilable circuits, it is called polycyclic. When it is cyclic for only one set of circuits, it is called monocyclic.

EXAMPLE.—The apparent area of a circle as seen from a point ($x$, $y$, $z$) anywhere in space, is a monocyclic function of $x$, $y$, $z$, of which the cyclic constant is $4\pi$.

The apparent area of a plane curve of the ($2n$)th degree, consisting of $n$ detached closed (that is finite endless) branches (some of which might be enclosed within others) is an $n$-cyclic function, of which the $n$ cyclic constants are essentially equal, being each $4\pi$.

Algebraic equations among three variables ($x$, $y$, $z$), may easily be found to represent tortuous curves, constituting one or more finite, isolated, endless branches (which may be knotted, as shown in the first three diagrams of § 58, or linked into one another, as in the fourth and fifth). The integral expressing what, for brevity, we shall call the *apparent area* of such a curve, is a cyclic function, which, if polycyclic, has essentially equal values for all its cyclic constants. By the *apparent area of a finite endless curve* (tortuous or plane), I mean the *sum of the apparent areas of all barriers edged by it, which we can draw without making a partition.*

It is worthy of notice that every polycyclic function may be reduced to a sum of monocyclic functions.

($y$). Fluid motion is called *cyclic* unless the circulation is zero in every closed path through the fluid, when it is called *acyclic*. Rotational motion is ($e$) essentially cyclic.

($z$). Irrotational motion may [ § 59 ($f$)] be either acyclic or cyclic. If cyclic it is *monocyclic* if there is only one distinct circuit, or *polycyclic* if there are several distinct circuits, in which there is circulation. It is *purely cyclic* if the boundary of the space occupied by irrotationally moving fluid is at rest. If the boundary moves and the motion of the fluid is cyclic, it is *acyclic compounded with cyclic.*

61. ($a$). We are now prepared to investigate the most general possible irrotational motion of a single continuous fluid mass, occupying either simply or multiply continuous space, with for every point of the boundary a normal component velocity given arbitrarily, subject only to the condition that the whole volume remains unaltered.

($b$). *Genesis of acyclic motion.* Commencing, as in § 3, with a fluid mass at rest throughout, let all multiplicity of the continuity of the space occupied by it be done away with by temporary barrier surfaces, $\beta_1, \beta_2 \ldots$ stopping the circuits, as described in § 57. The bounding surface of the fluid, which ordinarily consists of the inner surface of the containing vessel, will thus be temporarily extended to include each side of each of these barriers. Let now, as in § 3, any possible motion be arbitrarily given to the bounding surface. The liquid is consequently set in motion, purely through fluid pressure; and the motion is [§§ 10–15, or 60, 59] throughout irrotational. Hence irrotational motion fulfilling the prescribed surface conditions is possible, and the actual motion is, of course (as the solution of every real problem is), unambiguous. But from this bare physical principle we could not even suspect, what the following simple application of GREEN's equation proves, that the surface normal velocity at any instant determines the interior motion irrespectively of the previous history of the motion from rest.

61. (*c*). *Determinacy of irrotational motion in simply continuous space.* In § 57 (1), which is immediately applicable, as the volume is now simply continuous, make $\varphi' = \varphi$, and put $\nabla^2\varphi = 0$, so that $\varphi$ may be the velocity potential of an incompressible fluid. That double equation becomes the following single equation—

$$\iiint\left(\frac{d\varphi^2}{dx^2} + \frac{d\varphi^2}{dy^2} + \frac{d\varphi^2}{dz^2}\right)dx\,dy\,dz = \iint d\sigma\varphi\mathfrak{d}\varphi\,,$$

where the surface integration $\iint d\sigma$ must now include each side of each of the barrier surfaces $\beta_1$, $\beta_2$ .... Hence, if $\mathfrak{d}\varphi = 0$ for every point of the bounding surface, we must have

$$\iiint\left(\frac{d\varphi^2}{dx^2} + \frac{d\varphi^2}{dy^2} + \frac{d\varphi^2}{dz^2}\right)dx\,dy\,dz = 0\,,$$

which requires that

$$\frac{d\varphi}{dx} = 0\,,\quad \frac{d\varphi}{dy} = 0\,,\quad \frac{d\varphi}{dz} = 0:$$

that is to say, if there is no motion of the boundary surface in the direction of the normal, there can be no motion of the irrotational species in the interior; whence it follows that there cannot be two different internal irrotational motions with the same surface normal component velocities. Thus, as a particular case, beginning with a fluid at rest, let its boundary be set in motion; and brought again to rest at any instant, after having been changed in shape to any extent, through any series of motions. The whole liquid comes to rest at that instant.

A demonstration of this important theorem, which differs essentially from the preceding, and includes what the preceding does not include, a purely analytical proof of the possibility of irrotational motion throughout the fluid, fulfilling the arbitrary surface-condition specified above, was first published in THOMSON and TAIT's "Natural Philosophy," § 317 (3), and is to be given below, with some variation and extension. In the meantime, however, we satisfy ourselves as to the *possibility* of irrotational motions fulfilling the various surface-conditions with which we are concerned, because the surface motions are possible and require the fluid to move, and [§§ 10–15, or § 59] because the fluid cannot acquire rotational motion through fluid pressure from the motion of its boundary; and we go on, by aid of GREEN's extended formula [§ 57 (7)], to prove the determinateness of the interior motion under conditions now to be specified for multiply continuous space, as we have done by his unaltered formula [§ 57 (1)] for simply continuous space.

62. *Genesis of Cyclic Irrotational Motion.*—In the case of motion considered in § 61, the value of the normal component velocity is not independently arbitrary over the whole boundary, but has equal arbitrary values, positive and negative, on the two sides of each of the barriers $\beta_1$, $\beta_2$, &c. We must now introduce a fresh restriction in order that, when the barriers are liquefied, the motion of the fluid may be irrotational throughout the space thus re-opened into multiple continuity. For although we have secured that the normal component velocity is equal everywhere on the two sides of each barrier, we have hitherto left the tangential velocity unheeded. If they are not equal on the two sides, and in the same direction, there will be a finite slipping of fluid on fluid across the surface left by the dissolution of the infinitely thin barrier membrane; constituting [§ 60 ($m$) above], as HELMHOLTZ has shown, a "vortex sheet." The analytical expression of the condition of equality between the tangential velocities is that the variation of the velocity potential in tangential directions shall be equal on the two sides of each barrier. Hence, by integration, we see that the difference between the values of the velocity potential on the two sides must be the same over the whole of each barrier. This condition requires that the initiating pressure be equal over the whole membrane. For, at any time during the instituting of the motion, let $p_1$, $p_2$ be the pressures at two points $P_1$, $P_2$ of the fluid, and moving with the fluid, infinitely near one another on the two sides of one of the membranes, so that the pressure $\varpi$, which must be applied to the membrane to produce this difference of fluid pressure on the two sides, is equal to $p_1 - p_2$ in the direction opposed to $p_1$. And let $\varphi_1$, $\varphi_2$ be the velocity potentials at $P_1$ and $P_2$, so that if $\int ds$ denote integration from $P_1$ to $P_2$, along any path $P_1PP_2$ whatever from $P_1$ to $P_2$, altogether through the fluid (and therefore cutting none of the membranes), and $\zeta$ the component of fluid velocity along the tangent at any point of this curve, we have

$$\int \zeta ds = \varphi_2 - \varphi_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Hence, by (6) of § 59,

$$\frac{d(\varphi_2 - \varphi_1)}{dt}, = \varpi - \tfrac{1}{2}(q_1^2 - q_2^2) \quad . \quad . \quad . \quad . \quad (2),$$

where $q_1$, $q_2$ denote the resultant fluid velocities at $P_1$ and $P_2$. Now, the normal component velocities at $P_1$ and $P_2$ are necessarily equal; and therefore, if the components parallel to the tangent plane of the intervening membrane are also equal, we have

$$q_1 = q_2$$

and the preceding becomes

$$\frac{d(\varphi_2 - \varphi_1)}{dt} = \varpi \quad . \quad . \quad . \quad . \quad . \quad (3).$$

But if the tangential component velocities at $P_1$ and $P_2$ are not only equal, but in the same direction, $\varphi_2 - \varphi_1$ must, as we have seen, be constant over the membrane, and therefore $\varpi$ must also be constant.

Suppose now that after pressure has been applied for any time in the manner described, of uniform value all over the membrane at each instant, it is applied no longer, and the membrane (having no longer any influence) is done away with. The fluid mass is left for ever after in a state of motion, which is irrotational throughout, but cyclic. The "circulation" [§ 60 ($a$)], or the cyclic constant being equal to $\varphi_2 - \varphi_1$, for every circuit reconcilable with $P_1PP_2P_1$ is given by the equation

$$\varphi_2 - \varphi_1 = -\int \varpi dt \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

$\int dt$ denoting a time-integral extended through the whole period during which $\varpi$ had any finite value.

The same kind of operation may be performed, on each of the $n$ barriers temporarily introduced in § 61 to reduce the $(n+1)$fold continuity of the space occupied by the fluid, to simple continuity.

The velocity potential at any point of the fluid will then be a polycyclic function [§ 60 ($x$)] equal to the sum of the separate values corresponding to the pressure separately applied to the several barriers. Thus we see how a state of irrotational motion, cyclic with reference to every one of the different circuits of a multiply continuous space, and having arbitrary values for the corresponding cyclic constants, or circulations, may be generated. But the proof of the possibility of fluid motion fulfilling such conditions, founded on this planning out of a genesis of it, leaves us to imagine that it might be different according to the infinitely varied choice we may make of surfaces for the initial forms of the barriers, or according to the order and the duration of the applications of pressure to them in virtue of which these figures may be changed more or less, and in various ways, before the initiating pressures all cease; and hitherto we have seen no reason even to suspect the following proposition to the contrary.

63. (Prop.) The motion of a liquid moving irrotationally within an $(n+1)$ply continuous space is determinate when the normal velocity at every point of the boundary, and the values of the circulations in the $n$ circuits, are given.

This is proved by an application of Green's extended formula (7) of § 57, showing, as the simple formula (1) of the same section showed us in § 61 for simply continuous space, that the difference of the velocity potentials of two motions, each fulfilling this condition, is necessarily zero throughout the whole fluid. Let $\varphi, \varphi'$ be the velocity potentials of two motions fulfilling the prescribed conditions, and let

$$\psi = \varphi - \varphi'.$$

At every point of the boundary (the barriers not included) the prescribed conditions require that $\mathfrak{d}\varphi = \mathfrak{d}\varphi'$, and therefore $\mathfrak{d}\psi = 0$. Again, the cyclic constants for $\varphi'$ are equal to those for $\varphi$; those for $\psi$, being their differences, must therefore vanish. Hence, if the $\varphi$ and $\varphi'$ of § 57 (7) be made equal to one another and to avoid confusion with our present notation we substitute $\psi$ for each, the second members of that double equation vanish, and it becomes simply

$$\iiint \left( \frac{d\psi^2}{dx^2} + \frac{d\psi^2}{dy^2} + \frac{d\psi^2}{dz^2} \right) dx\, dy\, dz = 0\,;$$

which, as before (§ 61), proves that $\psi = 0$, and therefore $\varphi' = \varphi$; and so establishes our present proposition.

EXAMPLE (1). The solution $\varphi = \tan^{-1}\frac{y}{x}$ considered in § 56, fulfils LAPLACE's equation, $\nabla^2\varphi = 0$; and obviously satisfies the surface condition, not merely for the annular space with rectangular meridional section there considered, but for the hollow space bounded by the figure of revolution obtained by carrying a closed curve of any shape round any axis (OZ) not cutting the curve; which, for brevity, we shall in future call a *hollow circular ring*. Hence the irrotational motion possible within a fixed hollow circular ring is such that the velocity potential is proportional to the angle between the meridian plane through any point, and a fixed meridian.

EXAMPLE (2). The solid angle, $\alpha$, subtended at any point $(x, y, z)$, by an infinitesimal plane area, A, in any fixed position, fulfils LAPLACE's equation $\nabla^2\alpha = 0$. This well-known proposition may be proved by taking A at the origin, and perpendicular to OX, when we have

$$\alpha = \frac{\mathrm{A}x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} = \mathrm{A}\frac{d}{dz}\frac{-1}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} \quad . \quad . \quad (5),$$

for which $\nabla^2\alpha = 0$ is verified.

The solid angle subtended at $(x, y, z)$ by any single closed circuit is the sum of those subtended at the same point by all parts into which we may divide any limited surface having this curve for its bounding edge. [Consider particularly curves such as those represented by the first three diagrams of § 58.] Hence if we call $\phi$ the solid angle subtended at $(x, y, z)$ by this surface, LAPLACE's equa- $\nabla^2\phi$ is fulfilled. Hence $\phi$ represents the velocity potential of the irrotational motion possible for a liquid contained in an infinite fixed closed vessel, within which is fixed, at an infinite distance from the outer bounding surface, an infinitely thin wire bent into the form of the closed curve in question.

The particular case of this example for which the curve is a circle, presents us with the simplest specimen of cyclic irrotational motion not confined [as that of Example (1) is] to a set of parallel planes. The velocity potential being the apparent area of a circular disc (or the area of a spherical ellipse) is readily found, and shown to be expressible readily in terms of a complete elliptic integral of the third class, and therefore in terms of incomplete elliptic functions of the first and second classes. The equi-potential surfaces are therefore traceable by aid of LEGENDRE'S tables. But it is to HELMHOLTZ that we owe the remarkable and useful discovery, that the equations of the *stream lines* (or lines perpendicular to the equi-potential surfaces) are expressible in terms of complete integrals of the first and second classes. They are therefore easily traceable by aid of LEGENDRE'S tables. The annexed diagram, of which we shall make much use later, show these curves as calculated and drawn by Mr MACFARLANE from HELMHOLTZ'S formula, expressed in terms of rectangular co-ordinates. An improved method of tracing them is described in a note by Mr CLERK MAXWELL, which he has kindly allowed me to append to this paper.

EXAMPLE 3. The motion described in Example 2 will remain unchanged outside any solid ring formed by solidifying and reducing to rest a portion of the fluid bounded by stream lines surrounding the infinitely thin wire. Thus we have a solid thick endless wire or bar forming a ring, or an endless knot as illustrated in the first three diagrams of § 59, of peculiar sectional figure depending on the stream lines round the arbitrary curve of Example 2; and the cyclic irrotational motion which, if placed in an infinite liquid it permits, is that whose velocity potential is proportional to the solid angle defined geometrically in the general solution given under Example 2.

64. *Kinetic energy of compounded acyclic and polycyclic irrotational motion—kinetico-statics.* The work done in the operation described in § 62 is calculated directly by summing the products of the pressure into an infinitesimal area of the surface, into the space through which the fluid contiguous with this area moves in the direction of the normal, for all parts of the surface, whether boundary or internal barrier, where the genetic pressure is applied, and for all infinitesimal divisions of the whole time from the commencement of the motion.

(*a*). Let $w$ denote the work done, and $\int dt$ time-integration, from the beginning of motion up to any instant. At any previous instant let $p$ be the pressure, $q$ the velocity, and $\phi$ the velocity potential, of the fluid contiguous to any element $d\sigma$ of the bounding surface, $k$ the difference of fluid pressures on the two sides of any element, $d\varsigma$, of one of the internal barriers, and N the normal component of the fluid velocity contiguous to either $d\sigma$ or $d\varsigma$. The preceding statement expressed in symbols is

$$W = \int dt\left[-\iint p\mathrm{N}d\sigma + \Sigma\iint k\mathrm{N}d\varsigma\right] \quad . \quad . \quad . \quad . \quad (6),$$

$\Sigma$ denoting summation for the several barriers if there are more than one. According to the general hydrokinetic theorem for irrotational motion [§ 59 (6) compare with § 31 (5)], *with $\varphi$ expressed in terms of the co-ordinates of a point moving with the fluid*, we have

$$p = -\frac{d\varphi}{dt} + \tfrac{1}{2}q^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Now, let us suppose the pressure to be impulsive, so that there is infinitely little change of shape either of the bounding surface or of the barriers during the time $\int dt$. This will also imply that $\frac{d\phi}{dt}$ is infinitely great in comparison with $\frac{1}{2}q^2$; so that

$$p = -\frac{d\varphi}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

And according to the notation of § 57 we have

$$\mathrm{N} = \mathfrak{d}\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

Also $k$ is constant over each barrier surface.
Hence (6) becomes

$$\mathrm{W} = \int dt \left[ \iint \frac{d\varphi}{dt} \mathfrak{d}\varphi d\sigma + \Sigma k \iint \mathfrak{d}\varphi d\varsigma \right] \quad . \quad . \quad . \quad (10).$$

64. (*b*). The initiating motion of the bounding surface and the pressures on the barriers may be varied quite arbitrarily from the beginning to the end of the impulse; so that the history within that period of the acquisition of the prescribed final velocity may be altogether different, and not even simultaneous, in different parts of the bounding surface. Thus $k_1$ and $k_2$ may be quite different functions of $t$; provided only $\int k_1 dt$ and $\int k_2 dt$ have the prescribed values, which we shall denote by $\mathfrak{k}_1$ and $\mathfrak{k}_2$ respectively.

(*c*). But, for one example, we may suppose $\varphi$ to have at each instant of $\int dt$ everywhere one and the same proportion of its final value; so that if the latter denoted by $\Phi$, and if we put

$$\frac{\varphi}{\Phi} = m \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

$m$ is independent of co-ordinates of position, but may of course be any arbitrary function of the time. Hence, observing that

$$\int dt\, m \frac{dm}{dt} = \tfrac{1}{2},$$

as the final value of $m$ is 1, (10) becomes

$$\mathrm{W} = \tfrac{1}{2}[\iint \Phi \mathfrak{d}\Phi d\sigma + \Sigma \mathfrak{k} \iint \mathfrak{d}\Phi d\varsigma] \quad . \quad . \quad . \quad . \quad (12).$$

(*d*). The second member of this equation doubled agrees with the two equal second members of (7) § 57 with $\varphi$ and $\varphi'$ each made equal to $\Phi$. And the first member of that equation becomes twice the kinetic energy of the whole motion. Hence, when $\varphi'=\varphi$, and $\nabla^2\varphi=0$, (7) of § 57 expresses the equation of energy for the impulsive generation, of the fluid motion corresponding to velocity potential $\varphi$, by pressures varying throughout according to the same function of the time; the first member being twice the kinetic energy of the motion generated, and the second twice the work done in the process.

64. (*e*). As another example, let us suppose the initiating pressures to be so applied as first to generate a motion corresponding to velocity potential $\varphi$, and after that to change the velocity potential from $\varphi$ to $\varphi+\varphi'$, denoting by $\varphi$ and $\varphi'$ any two functions, such that $\varphi+\varphi'=\Phi'$, and each fulfilling LAPLACE's equation: and let the augmentation from zero to $\varphi$, and again from $\varphi$ to $\varphi+\varphi'$ be uniform through the whole fluid. The work done in the first process, found as above (12),

$$\tfrac{1}{2}[\iint\varphi\mathfrak{D}\varphi\, d\sigma+\Sigma\kappa\iint\mathfrak{D}\varphi\, d\varsigma] \quad . \quad . \quad . \quad . \quad (13),$$

if $\kappa_1$, $\kappa_2$, &c., denote the cyclic constants relative to $\varphi$, as $\mathfrak{K}_1$, $\mathfrak{K}_2$, &c., relatively to $\Phi$, and the additional work done in the second process, similarly found, is

$$\tfrac{1}{2}[\iint\varphi'(2\mathfrak{D}\varphi+\mathfrak{D}\varphi')\, d\sigma+\Sigma\kappa'\iint(2\mathfrak{D}\varphi+\mathfrak{D}\varphi')\, d\varsigma] \quad . \quad . \quad . \quad (14).$$

(*f*). Now, as we have seen (§ 63) that the actual fluid motion depends at each instant wholly on the normal velocity at each point of the bounding surface and the values of the cyclic constants, it follows that the work done in generating it ought to be independent of the order and law, of the acquisition of velocity at the bounding surface, and of the attainment of the values of the several cyclic constants. Hence, the the sum of (13) and (14) ought to be equal to (12). But if, for $\Phi$ in (12) we substitute $\varphi+\varphi'$, the difference between its value and that of the sum of (13) and (14) is found to be

$$\tfrac{1}{2}[\iint(\varphi\mathfrak{D}\varphi'-\varphi'\mathfrak{D}\varphi)\, d\sigma+\Sigma(\kappa\iint\mathfrak{D}\varphi' d\varsigma-\kappa'\iint\mathfrak{D}\varphi d\varsigma)] \quad . \quad . \quad . \quad (15);$$

which, being the half the difference between the two equal second members of (7) § 57 for the case of

$$\nabla^2\varphi=0 \quad \text{and} \quad \nabla^2\varphi'=0\,,$$

is equal to zero. Hence, the equality of the second members of (7) § 57, constitutes the analytical reconciliation of the equations of energy for different modes of generation of the same fluid motion.