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I. *On Travelling Atmospheric Disturbances.*

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IT has been shown by Lamb † that an arbitrary disturbance of the uniform distribution of density in the atmosphere in horizontal layers would ordinarily give rise to a motion of the nature of a wave spreading out from the originally disturbed region, the velocity of propagation being of the same order of magnitude as that of sound. Thus a local variation in mass distribution would be rapidly dispersed over a wide area and the original uniform state restored. Any disturbance of even moderately permanent character must therefore be of a very special type. Now the ordinary cyclone is able to retain its size and pressure distribution about its centre for days, the velocity with which it moves being of the order of twenty feet per second, very much smaller than the speed of an atmospheric wave. Thus the first question with regard to the nature of a cyclone is, why does it not spread out like an ordinary wave and disperse in an hour or two? The answer seems to be that a cyclone is of the character of a *standing wave*, the pressure and velocity distribution being such that for this peculiar kind of disturbance the velocity of propagation is practically zero. The realization of this fact has led to the assumption of the well-known "Gradient relation," according to which the

* Communicated by the Author.

† "On Atmospheric Oscillations," Proc. Roy. Soc. lxxxiv. A. pp. 551-572 (1910).

wave velocity is very small, or, what is equivalent, that the acceleration terms in the equations of motion of the air can be neglected. If (u, v, w) be the components of velocity of the air at the point (x, y, z), at time t, the axis of z being vertical, the equations of motion take the form *

$$\left.\begin{aligned} \frac{du}{dt}-2\omega v &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ \frac{dv}{dt}+2\omega u &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \\ \frac{dw}{dt} \quad &= -g-\frac{1}{\rho}\frac{dp}{dz} \end{aligned}\right\}, \quad \ldots \quad (1)$$

where
$$\frac{d}{dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z},$$

p is the pressure,
ρ is the density,
g is the acceleration due to gravity,
and ω is the component about the vertical of the earth's angular velocity of rotation.

Friction, which is only effective in the surface-layers, is ignored.

It is assumed first that the vertical velocity can be neglected. The writer has shown elsewhere that † this is justifiable in problems of winds caused by temperature variations of horizontal extent large compared with the height of the atmosphere, and as an average cyclone is of the order of 1000 km. across the same is probably true here. The assumption must not, however, be pushed too far; it would be in serious error for small islands, and probably also for land and sea breezes of the usual diurnal type. For the ordinary widespread depression, however, it is probably correct, and the third equation of motion becomes

$$p=\int_z^\infty g\rho dz. \quad \ldots \quad (2)$$

The assumption that the disturbance is permanent gives at once that $\frac{\partial u}{\partial t}$ and $\frac{\partial v}{\partial t}$ are zero; hence the two equations of horizontal motion become

$$\left.\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-2\omega v &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+2\omega u &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \end{aligned}\right\} \quad \ldots \quad (3)$$

* *Cf.* Lamb, 'Hydrodynamics,' p. 302.
† Phil. Mag. vol. xxxiv. pp. 449–458 (1917).

Further simplification is not possible without some knowledge of the size of the quantities involved. If it be assumed, as is usually correct in these latitudes, that the pressure gradients and wind velocities are small enough for their squares to be neglected in a first approximation, we have nearly

$$\left.\begin{aligned} -2\omega v &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ 2\omega u &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which gives the so-called "Geostrophic relation." If the isobars are concentric circles we have as a second approximation, the density being assumed constant, the equation

$$\frac{V^2}{r} + 2\omega V = \frac{1}{\rho}\frac{\partial p}{\partial r}, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where V is the resultant velocity and r the distance from the common centre, which is the usual form of the equation giving the wind in terms of the pressure gradient. It is often assumed to be correct even when the pressure distribution is changing with the time ; but this involves the assumption that the air is not accelerated along its path, which means that $\frac{dV}{dt}$ is neglected while $\frac{\partial p}{\partial t}$ is retained. This is a very uncertain hypothesis, for in general these two quantities would be expected to be of the same order of magnitude.

When the pressure distribution is varying, the problem becomes much more complex. A cyclone in most cases moves fairly steadily in one direction, the isobars remaining approximately concentric. Sometimes the depression in the centre deepens as it moves, more often it becomes shallower and spreads out, but frequently it travels for thousands of miles practically unchanged. Now if it were merely a wave free to spread out, it would, as was said before, do so with a velocity comparable with that of sound, and would therefore disappear in a few hours. On the other hand, the motion of the depression is not itself of the character of the propagation of a wave, for it only takes place at the rate of some feet per second. Thus the moving depression, like the stationary one, requires peculiar conditions for its maintenance, and it is intended to indicate some of these in the present paper. The method adopted is that of successive approximation according to powers of the pressure gradient.

Suppose that the speed of translation of the cyclone is small, of the same order of magnitude at least as that of the winds themselves. Then $\partial/\partial t$ is of the order $u\partial/\partial x$ and the only first order terms in the equations of motion are those in the equations

$$\left.\begin{aligned} -2\omega v &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ 2\omega u &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Thus the geostrophic relation still holds as a first approximation. When, however, the second powers of the velocities are negligible, no disturbance can travel. For the rate of increase of the surface pressure at any point is given by

$$\frac{\partial p_0}{\partial t} = \int_0^\infty g\frac{\partial \rho}{\partial t}dz = -g\int_0^\infty \left\{\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right\}dz, \quad (7)$$

by the equation of continuity.

Now ρw is zero when $z=0$ and when z is infinite, since there is no vertical velocity on the ground and the density tends to zero at a great height. Hence

$$\frac{\partial p_0}{\partial t} = -g\int_0^\infty \left\{\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v)\right\}dz. \quad . \quad . \quad (8)$$

Substituting in this from the equations (6) we have identically

$$\frac{\partial p_0}{\partial t} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Thus to this order the pressure distribution is not varying. Hence if it is changing at all it must depend on powers of the pressure gradient higher than the first. The assumption that this kind of approximation is possible is therefore so far justified.

A second approximation may now be obtained by substituting the values of u and v given by (6) into the terms in (1) depending on the squares of the velocities, and again determining u and v as far as the second powers of the pressure gradients. As $\partial p/\partial t$ is of higher order than $u\partial p/\partial x$, it follows that $\partial u/\partial t$ and $\partial v/\partial t$ are of higher order than $u\partial u/\partial x$; thus they contain no terms of the second order and therefore must be neglected. The vertical velocity also must be zero unless the distribution is changing; hence to this order it also may be neglected in the equations of motion. Then

$$\left.\begin{aligned} 2\omega\rho u &= -\frac{\partial p}{\partial y} + \frac{1}{2\omega}\left(\frac{\partial p}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial v}{\partial y}\right) \\ 2\omega\rho v &= \frac{\partial p}{\partial x} - \frac{1}{2\omega}\left(\frac{\partial p}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial u}{\partial y}\right) \end{aligned}\right\}, \quad . \quad (10)$$

and

$$\frac{\partial p_0}{dt} = -\frac{g}{4\omega^2}\int_0^\infty \left\{\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial p}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial u}{\partial y}\right)\right\} dz, \quad (11)$$

where for u and v we must substitute their values from (4).

So far these results are general, subject to the validity of the approximations made, which seems satisfactory in ordinary cases. Further progress, however, requires some knowledge of the relations connecting pressure and density with position; and when the actual laws that hold in the atmosphere are substituted the formulæ soon become unmanageable. An approximation can nevertheless be employed that enables the actual conditions to be imitated without making the algebra quite intractable. In the ordinary cyclone below the stratosphere the difference of pressure from normal does not vary greatly with the height, and it appears as if the disturbance arose from a change within the stratosphere: thus the conditions within the troposphere could be represented by a variation in the height of the free surface of an incompressible fluid. At the same time there is a well marked temperature gradient, the temperature usually increasing towards the south in the troposphere; the opposite seems to hold in the stratosphere*. Thus the troposphere can be represented by an incompressible fluid of finite depth whose temperature and therefore whose density are monotonic functions of one of the horizontal coordinates. Let the mean height of the free surface be H, and the excess of the actual height of any point of the free surface above this be ζ. Also let the density be given by $\rho = \rho_0 + \rho_1$, where ρ_1 is small and a function of x and z only.

It is further supposed that ζ and ρ_1 are small enough for

* W. J. Humphreys, Bull. Mt. Weather Obs. vol. ii. pp. 292–297 (1910).

their third powers to be neglected in the present approximation. Then

$$u=-\frac{g}{2\omega}\frac{\partial\zeta}{\partial y};\ v=\frac{g}{2\omega}\frac{\partial\zeta}{\partial x}+\frac{g}{2\omega\rho_0}\int_z^{\mathrm{H}}\frac{\partial\rho_1}{\partial x}dz\ \ldots \quad (12)$$

to the first order, and substituting from these into (11) we find

$$-\frac{8\omega^3}{g^3\rho_0}\frac{\partial\rho_0}{\partial t}=\left(\frac{\partial\zeta}{\partial y}\frac{\partial^3\zeta}{\partial x^3}-\frac{\partial\zeta}{\partial x}\frac{\partial^3\zeta}{\partial x^2\partial y}-\frac{\partial\zeta}{\partial x}\frac{\partial^3\zeta}{\partial y^3}+\frac{\partial\zeta}{\partial y}\frac{\partial^3\zeta}{\partial x\,\partial y^2}\right)\mathrm{H}$$

$$-\left(\frac{\partial^3\zeta}{\partial x^2\partial y}+\frac{\partial^3\zeta}{\partial y^3}\right)\frac{1}{\rho_0}\int_0^{\mathrm{H}}\int_z^{\mathrm{H}}\frac{\partial\rho_1}{\partial x}dz+\frac{1}{\rho_0}\frac{\partial\zeta}{\partial y}\int_0^{\mathrm{H}}\int_z^{\mathrm{H}}\frac{\partial^3\rho_1}{\partial x^3}dz\,dz$$

$$=\mathrm{H}\left(\frac{\partial\zeta}{\partial y}\frac{\partial}{\partial x}\nabla_1^2\zeta-\frac{\partial\zeta}{\partial x}\frac{\partial}{\partial y}\nabla_1^2\zeta\right)$$

$$+\frac{1}{\rho_0}\frac{\partial\zeta}{\partial y}\int_0^{\mathrm{H}}\int_z^{\mathrm{H}}\frac{\partial^3\rho_1}{\partial x^3}dz\,dz-\frac{1}{\rho_0}\frac{\partial}{\partial y}\nabla_1^2\zeta\int_0^{\mathrm{H}}\int_z^{\mathrm{H}}\frac{\partial\rho_1}{\partial x}dz\,dz$$

$$=\frac{\mathrm{H}\partial(\nabla_1^2\zeta',\zeta')}{\partial(x,y)}\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ . \quad (13)$$

where
$$\zeta'=\zeta+\frac{1}{\rho_0\mathrm{H}}\int_0^{\mathrm{H}}\int_z^{\mathrm{H}}\rho_1\,dz\,dz,\ .\ .\ .\ . \quad (14)$$

and
$$\nabla_1^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

Now $g\rho_0\zeta'$ is the mean of the excesses of the pressure above normal at all points of a vertical column. Also by hypothesis $\partial\rho/\partial t=0$, and therefore if $\mathrm{P}=g\rho_0\zeta'$, P satisfies the equation

$$-\frac{8\omega^3\rho_0}{g\mathrm{H}}\frac{\partial\mathrm{P}}{\partial t}=\frac{\partial(\nabla_1^2\mathrm{P},\mathrm{P})}{\partial(x,y)}.\ \ .\ .\ .\ . \quad (15)$$

From this several interesting consequences can be deduced at once. If for instance this mean pressure anomaly is constant over concentric circles, it is a function of the distance from the common centre of these circles, and therefore $\nabla_1^2\mathrm{P}$ is a function of P and the Jacobian vanishes. Thus if a depression is perfectly symmetrical the pressure cannot vary with the time, and therefore all pressure changes must be caused by departures from circular symmetry. Again, if the curves P=constant are symmetrical with respect to two perpendicular axes, let us take these to be the axes of x and y; then P is an even function of x and y, and therefore so is $\nabla_1^2\mathrm{P}$; hence the Jacobian is an even function of x and y and the same must apply to the pressure changes.

Motion of the depression in one definite direction cannot therefore occur, though it may change its form subject to the condition that it must remain symmetrical with respect to these axes. Hence a cyclone cannot travel if it is symmetrical with respect to two axes; if it is to do so it must be definitely ovoid in form. A case of motion produced by asymmetry is afforded by the ordinary cyclone. This forms at the edge of a region of low pressure, so we shall take the general distribution of pressure to be decreasing in the direction of x increasing. In such a system a depression forms, the lowest pressure being near its centre. It seems, however, that the distribution does not consist of a symmetrical distribution on which a uniform increase in one direction is superposed. For if it were so, we should have

$$P = -Bx + R,$$

where R is a function of x^2+y^2 only, and B is a constant. Then

$$\frac{\partial(\nabla_1^2 P, P)}{\partial(x, y)} = B\frac{\partial \nabla_1^2 R}{\partial y},$$

and the rate of variation of P is given by

$$\frac{\partial P}{\partial t} = -\frac{gHB}{8\omega^3\rho_0}\frac{\partial \nabla_1^2 R}{\partial y}.$$

Now if the disturbance is travelling unaltered we must have

$$\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y} = 0,$$

where U and V are the components of the velocity of translation, and the object of the operation is any of the physical quantities of the system. But the density does not depend on t or y, and therefore this operation on it shows that U is zero. Next applying it to P we find

$$V\frac{\partial R}{\partial y} - \frac{gHB}{8\omega^3\rho_0}\frac{\partial \nabla_1^2 R}{\partial y} = 0.$$

First, consider the order of magnitude of the velocity of translation thus indicated. Let the horizontal dimensions of the cyclone be of order a; then $\nabla_1^2 R$ is comparable with R/a^2, and V is of order $-\dfrac{gHB}{8\omega^3\rho_0 a^2}$.

Now $B/2\omega\rho_0$ is the geostrophic wind corresponding to the general pressure gradient; and $gH/4\omega^2$ is the square of the distance sound would travel in an interval comparable with 12 hours, which is itself much greater than the linear dimensions of an ordinary cyclone, as has already been mentioned. Thus if the pressure were distributed in the simple way here supposed, the velocity of translation would be much greater than the general velocity of the winds around the cyclone. This does not appear to be the case; for it would imply that the depression moved at such a rate that no part of the surrounding air could keep pace with it, and probably none of the air it itself contains. This contradicts the fact of the existence of a tornado centre *, or portion of air within the cyclone moving with the cyclone as a whole. The explanation seems to be that the actual effect of superposing a general pressure gradient on a mass of rotating air is to cause a considerably smaller change in its symmetry than would be expected, the reason being that some internal compensation reduces the asymmetry. That this reduction is not complete is, on the other hand, seen from the fact that the cyclone does move, and usually does so in the direction of the general gradient wind.

Summary.

The geostrophic relation between the wind and the surface pressure gradient is incapable of accounting for any variation whatever with time in the pressure distribution. All changes in this arise from those terms in the equations of motion that are neglected when the geostrophic relation is assumed. When these terms, which depend on the squares and differential coefficients of the velocities, are taken into account, it is found that an asymmetrical cyclone can move. It seems, however, from the low speed of travel of these depressions, that a simple superposition of a general pressure gradient on a rotating system must be compensated internally in some way, so as to reduce the asymmetry introduced. Thus the remarkable circularity of the isobars in a cyclone is seen to be a condition of its slow movement. It is indicated that the cyclone itself is a very special type of disturbance, in which the pressure, temperature, and velocity are so distributed as to make the wave tending to readjust it travel with extreme slowness; other types of disturbance spread out much more rapidly (with velocities of the order of that of sound) and are dissipated, and this fact is probably the reason why of all the irregularities possible the cyclone is the most conspicuous, other forms dissipating before they can be observed.

* Sir Napier Shaw, Geophysical Memoirs of the Meteorological Office, No. 12.