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SLOW THERMALLY OR FRICTIONALLY CONTROLLED MERIDIONAL CIRCULATION IN A CIRCULAR VORTEX

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Abstract: A quasi-static theory of meridional motion in a stable circular vortex, caused by sources of heat or angular momentum, is developed. The stream function of the meridional motion is found to satisfy a generalized Poisson equation in the meridional plane. In the vicinity of a point-source of angular momentum, the field of meridional motion is shown to have the character of a distorted field of a dipole, with elliptical streamlines. It is found that the meridional motion will take place mainly in the directions where the stability of the vortex is smallest, and that the speed of the meridional circulation, for given sources of heat or angular momentum, will increase with decreasing stability of the vortex. The accompanying changes in the kinetic energy of the vortex motion, and of the vortex structure, are discussed. An attempt is made to apply the theory to possible meridional currents in the earth's atmosphere.

Preface.

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1. Circular vortex in the state of balance.

The system under consideration in this study is a compressible fluid performing a circular vortex motion in a gravity field. The speed of rotation, the thermodynamical state of the fluid particles and the gravity potential are supposed to be constant along each circular streamline, so that the vortex is symmetric with respect to its axis. It will then be sufficient to consider the conditions in one meridional plane. The vortex motion will remain stationary as long as no friction is operating, and no heat is added to, or withdrawn from the fluid particles. Such

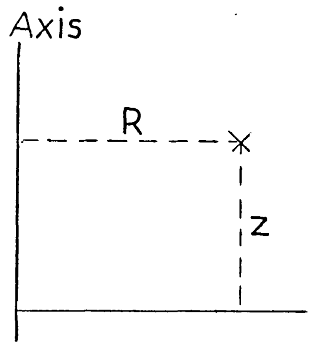

Fig. 1. Coordinates in the meridional plane.

a stationary circular vortex must be characterized by a balance between the force of gravity, the pressure force and the centrifugal force. We shall express this balance mathematically.

In a meridional plane, let R denote distance from the axis, and z height above a straight line drawn perpendicular to the axis (Fig. 1). Furthermore, let Φ denote gravity potential, p pressure and α specific volume. The field of motion may be characterized either by the absolute angular velocity ω of the motion around the axis, or by the angular momentum per unit mass (or circulation per radian of a circular streamline),

$$c = \omega R^2. \tag{1}$$

The quantities Φ, p, α, and c are constant for each circular streamline, but are supposed to vary from one circular streamline to another. These quantities may therefore be regarded as functions of R and z, and their fields may be represented by lines Φ = constant, p = constant, α = constant and c = constant drawn in a meridional plane.

With the above notation, the centripetal acceleration is c^2/R^3, so that the equations expressing the balance of the forces in the direction parallel to, and normal to the axis, respectively, are

$$0 = -\left(\frac{\partial \Phi}{\partial z}\right)_R - \alpha\left(\frac{\partial p}{\partial z}\right)_R \tag{2}$$

$$-\frac{c^2}{R^3} = -\left(\frac{\partial \Phi}{\partial R}\right)_z - \alpha\left(\frac{\partial p}{\partial R}\right)_z. \tag{3}$$

Here the subscript denotes the quantity that is held constant during the differentiation.

It has been shown by Sutcliffe (1947) and independently by the author (1949) that it is often advantageous to use pressure instead of height as an indenpendent variable in the equations governing the motion of the atmosphere. A similar advantage is attained in the present case by using R and p as independent variables instead of R and z. This means that the dependent variables Φ, α and c are regarded as functions of R and p, so that we may speak of the fields of the dependent variables in the Rp-plane. It is clear that knowledge of the field of a quantity in the Rp-plane does not involve knowledge of the field of the same quantity in the Rz-plane, unless we know also the field of pressure in the Rz-plane.

The transformation into the independent variables R and p may be carried out simply by multiplying eqs. (2) and (3) by dz and dR, respectively, and adding. This gives

$$d\Phi = -\alpha\, dp + \frac{c^2}{R^3}\, dR. \tag{4}$$

Hence we obtain, when Φ is regarded as a function of R and p,

$$\left(\frac{\partial \Phi}{\partial p}\right)_R = -\alpha \tag{5}$$

$$\left(\frac{\partial \Phi}{\partial R}\right)_p = \frac{c^2}{R^3}, \tag{6}$$

which is the form assumed by the system (2), (3) when R and p are independent variables.

It should be emphasized that the derivative with respect to R in eq. (6) means change along the isobar (at constant p), whereas the derivatives with respect to R in eq. (3) mean changes normal to the axis (at constant z). Throughout the following, the partial derivatives will have the same meaning as in eqs. (5) and (6), since R and p will be used consistently as independent variables. Thus there can be no confusion as to the meaning of these symbols, and the subscripts will therefore be dropped.

Our coordinate system R, p fails in points where the isobars are parallel to the axis, i. e. in "equatorial" points. The occurrence of such points would require a special investigation of the validity of our equations. We shall not here concern ourselves with this difficulty, and will therefore have to exclude such "equatorial" points from our considerations.

Eqs. (5) and (6) show that the fields of α and c in the Rp-plane are known if we know the function Φ (R, p). Furthermore, it follows that the fields of α and c are not independent of each other. Elimination of Φ between (5) and (6) gives

$$(7) \qquad \frac{1}{R^3}\frac{\partial c^2}{\partial p} = -\frac{\partial \alpha}{\partial R},$$

showing that in the state of balance, the variation of angular momentum (or angular velocity), in the direction parallel to the axis is related to the variation of α along the isobars, i. e. to the baroclinity. In particular, it follows that in a barotropic vortex, where α does not change along the isobars, the angular velocity is constant along each line parallel to the axis. These results have been derived by V. Bjerknes (1923).

It will be assumed in the following that the thermodynamic state of the fluid is characterized by two independent quantities. Taking pressure and entropy per unit mass, σ, to characterize the state, we may write

$$(8) \qquad \alpha = \alpha\ (p, \sigma).$$

The partial derivative of this function with respect to entropy will be denoted by

$$(9) \qquad \mu = \mu\ (p, \sigma) = \left(\frac{\partial \alpha}{\partial \sigma}\right)_p.$$

Usually μ is a positive quantity. The best known example of negative μ is water below $+4^\circ$ centigrade. For an ideal gas, $\mu = \alpha/c_p$, where c_p means specific heat at constant pressure.

On account of (9), eq. (7) may now be written

$$(10) \qquad \frac{1}{R^3}\frac{\partial c^2}{\partial p} = -\mu\frac{\partial \sigma}{\partial R},$$

since $\partial \alpha/\partial R$ means differentiation at constant pressure.

2. Quasi-static theory of slow meridional circulations.

Let us now assume that heat sources and sinks, and frictional forces, distributed symmetrically with respect to the axis of the vortex, are operating in the fluid. This will not destroy the axial symmetry of the vortex. The heat sources and sinks, together with the frictional dissipation, bring about a change with time of the entropy of the fluid particles; and the frictional forces will in general have a torque with respect to the axis, and thus cause a change with time of the angular momentum of the particles. As a result, the balance of the vortex will be disturbed, and meridional motions, superimposed upon the vortex motion, will take place.

The radiative and turbulent heat flux and the turbulent frictional stress depend upon the fields of motion and temperature within the vortex. Unfortunately, the manner in which the heat flux and the frictional stress depend upon the fields of motion and temperature is only poorely known. It is obviously of importance to see what conclusions can be drawn concerning the meridional motion without making use of these uncertain relationships. To this purpose, we shall in the following assume the sources of heat and angular momentum to be given, and investigate some properties of the resulting meridional motion.

The resulting meridional motion will have the character of a forced oscillation of a quite complicated type. We shall simplify the problem by assuming the sources of heat and angular momentum to be weak, and to change so slowly with time, that resonance phenomena will not occur. The resulting meridional currents may then be considered as being so slow that the accelerations due to these currents are small compared to the centripetal accelerations. The vortex will be very close to the state of balance all the time, so that we may apply the quasi-static approximation: we assume the vortex to be in the state of balance at all times, and determine the meridional motion necessary to maintain this balance. We shall see that the requirement of the maintenance of the balance is sufficient to determine the meridional motion uniquely.

Since the motion is no longer stationary, our dependent variables Φ, c, α, and σ must now be regarded as functions of the three independent variables R, p and time t. The quasi-static approximation means that the balance equations (5) and (6) are regarded as being valid for all values of t. These equations may therefore be differentiated partially with respect to t. Thus we obtain

(11) $$\frac{\partial}{\partial p}\frac{\partial \Phi}{\partial t} = -\frac{\partial \alpha}{\partial t} = -\mu\frac{\partial \sigma}{\partial t}$$

(12) $$\frac{\partial}{\partial R}\frac{\partial \Phi}{\partial t} = \frac{1}{R^3}\frac{\partial c^2}{\partial t}.$$

Here eq. (9) has been used to express $\partial\alpha/\partial t$ in terms of $\partial\sigma/\partial t$; this is possible because the symbol $\partial\alpha/\partial t$ means differentiation at constant R and p.

Denoting by Q the heat given to the fluid per unit mass and unit time, including the frictional dissipation, and by T the absolute temperature we have

(13) $$\frac{D\sigma}{dt} = \frac{Q}{T},$$

where D/dt means the individual (or substantial) derivative.

Furthermore, when the torque of the frictional force with respect to the axis is denoted by χ, we have

(14) $$\frac{Dc^2}{dt} = 2c\chi.$$

When R, p and t are independent variables, the individual derivative may be written as

(15) $$\frac{D}{dt} = \frac{\partial}{\partial t} + \dot{p}\frac{\partial}{\partial p} + \dot{R}\frac{\partial}{\partial R},$$

where $\dot{p}$ and $\dot{R}$ are the individual derivatives of p and R respectively. It will be seen that $\dot{p}$ and $\dot{R}$ represent a kind of meridional velocity components, so that the fields of $\dot{p}$ and $\dot{R}$ characterize the meridional circulation.

When the individual derivative in (13) and (14) are expanded in accordance with (15), we find

(16) $$\frac{\partial\sigma}{\partial t} + \dot{p}\frac{\partial\sigma}{\partial p} + \dot{R}\frac{\partial\sigma}{\partial R} = \frac{Q}{T}$$

(17) $$\frac{\partial c^2}{\partial t} + \dot{p}\frac{\partial c^2}{\partial p} + \dot{R}\frac{\partial c^2}{\partial R} = 2c\chi.$$

Eliminating $\partial\sigma/\partial t$ and $\partial c^2/\partial t$ between (11), (12), (16) and (17), we obtain the system

(18) $$-\frac{\partial}{\partial p}\frac{\partial \Phi}{\partial t} + \mu\frac{\partial\sigma}{\partial p}\dot{p} + \mu\frac{\partial\sigma}{\partial R}\dot{R} = \frac{\mu Q}{T}$$

(19) $$\frac{\partial}{\partial R}\frac{\partial \Phi}{\partial t} + \frac{1}{R^3}\frac{\partial c^2}{\partial p}\dot{p} + \frac{1}{R^3}\frac{\partial c^2}{\partial R}\dot{R} = \frac{2c\chi}{R^3}.$$

In addition to these two equations we will have to make use of the equation of continuity, which is derived below, in the coordinates R, p and t.

3. The equation of continuity.

Fig. 2 shows an infinitesimal parallellogram in the meridional plane, bounded by two isobars (p and $p + dp$) and two lines parallel to the axis (R and $R + dR$). The length of the sides parallel to the axis is

(20) $$dz = \left|\frac{\partial z}{\partial p}\right| dp = \frac{\alpha}{g_z} dp$$

according to equation (2). Here g_z means the component parallel to the axis of the acceleration of gravity.

We consider the ring-shaped volume formed by rotating the parallellogram around the axis. The mass contained within this volume is

(21) $$M = 2\pi R\,dR\,dz\frac{1}{\alpha} = \frac{2\pi R}{g_z}dR\,dp.$$

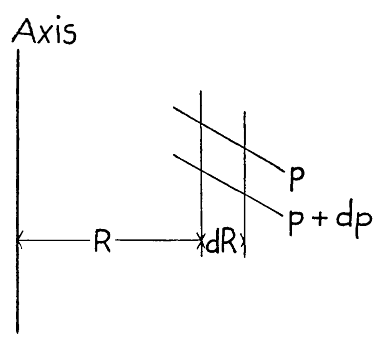

Fig. 2. Area element in the meridional plane.

Next we compute the flux of mass out of the same volume. The net flux out of the two cylindrical surfaces is seen to be $dR\, \partial/\partial R\; (2\,\pi\, R\, d\,z\, \dot{R}/\alpha)$ and the net flux out of the two isobaric surfaces $dp\, \partial/\partial p\, (2\,\pi\, R\, dR\, \dot{p}/g_z)$. The total flux out of the volume is thus

$$N = 2\,\pi\, R\, dR\, dp \left(\frac{1}{R}\frac{\partial}{\partial R}\frac{R\dot{R}}{g_z} + \frac{\partial}{\partial p}\frac{\dot{p}}{g_z}\right). \tag{22}$$

Consequently the equation of continuity is $N = -\,\partial M/\partial\, t$, or

$$\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{R\dot{R}}{g_z}\right) + \frac{\partial}{\partial p}\left(\frac{\dot{p}}{g_z}\right) = -\frac{\partial}{\partial t}\left(\frac{1}{g_z}\right). \tag{23}$$

Since $\partial/\partial\, t$ means differentiation at constant R and p, the right-hand side of this equation is seen to represent the rate of change of $(-1/g_z)$ experienced by an observer moving parallel to the axis with the speed of the isobars. In a stationary field of gravity, this term is therefore due solely to the motion of the isobars. In many cases, g_z will vary so slowly with z, and the motion of the isobars in the meridional plane will be so slow, that the right-hand side of (23) will be negligible. This is the case in the atmosphere, where the vertical velocity of the isobaric surfaces is of the order of magnitude 0.1—1 cm/sec. We will assume in the following that the right-hand term of (23) can be disregarded, so that the equation of continuity becomes

$$\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{R\dot{R}}{g_z}\right) + \frac{\partial}{\partial p}\left(\frac{\dot{p}}{g_z}\right) = 0\,. \tag{24}$$

We may satisfy this equation by writing

$$\dot{R} = \frac{g_z}{R}\frac{\partial\,\psi}{\partial p}, \quad \dot{p} = -\frac{g_z}{R}\frac{\partial\,\psi}{\partial R}, \tag{25}$$

where $\psi\,(R,\, p,\, t)$ is a kind of stream function for the meridional motion. The curves $\psi =$ constant are the streamlines of the meridional motion in the Rp-plane. These will generally differ from the streamlines in the Rz-plane. However, the difference will be slight when the motion of the isobaric surfaces is slow; and the streamlines in the Rp-plane will coincide with the streamlines in the Rz-plane if the pressure field is stationary.

4. The differential equation for the stream function.

Substitution of (25) into the system (18), (19) gives

$$-\frac{\partial}{\partial p}\frac{\partial \Phi}{\partial t}+A\frac{\partial \psi}{\partial R}+B\frac{\partial \psi}{\partial p}=E \tag{26}$$

$$\frac{\partial}{\partial R}\frac{\partial \Phi}{\partial t}+B\frac{\partial \psi}{\partial R}+C\frac{\partial \psi}{\partial p}=F, \tag{27}$$

where we have put

$$\left\{\begin{aligned} A &= -\frac{\mu g_z}{R}\frac{\partial \sigma}{\partial p} \\ B &= \frac{\mu g_z}{R}\frac{\partial \sigma}{\partial R} = -\frac{g_z}{R^4}\frac{\partial c^2}{\partial p}\ \text{(owing to eq. (10))} \\ C &= \frac{g_z}{R^4}\frac{\partial c^2}{\partial R} \\ E &= \frac{\mu Q}{T} \\ F &= \frac{2c\chi}{R^8}. \end{aligned}\right. \tag{28}$$

Regarding $\partial\Phi/\partial t$ and ψ as the unknown functions, eqs. (26) and (27) are seen to form a system of linear, first order differential equations in the two space coordinates R and p, whereas time drops out as an independent variable. In the quasi-static theory, the determination of the meridional motion is therefore not an initial value problem; the meridional motion depends only on the instantaneous sources of heat and angular momentum and the instantaneous structure of the vortex.

Eliminating $\partial\Phi/\partial t$ between (26) and (27), we obtain

$$\frac{\partial}{\partial R}\left(A\frac{\partial \psi}{\partial R}+B\frac{\partial \psi}{\partial p}\right)+\frac{\partial}{\partial p}\left(B\frac{\partial \psi}{\partial R}+C\frac{\partial \psi}{\partial p}\right)=\frac{\partial E}{\partial R}+\frac{\partial F}{\partial p} \tag{29}$$

which is the linear, second order differential equation satisfied by the stream function. It will be noted that the equation is self-adjoint.

The coefficients A, B and C of this equation appear also in the theory of axially symmetric oscillations of an adiabatic and frictionless circular vortex. This theory has been worked out by Solberg (1936), Høiland (1941), and others.[1] It follows from this theory that the frequency ν of oscillations with meridional streamlines parallel to a given direction defined by a certain value of dp/dR is given by

$$\nu^2 = \frac{R}{g_z} \sin^2 \varphi \left[A \left(\frac{dp}{dR} \right)^2 - 2 B \frac{dp}{dR} + C \right], \tag{30}$$

where φ is the angle in the meridional plane between the axis and the streamline direction. If we take the streamlines to be parallel to the axis, we obtain purely gravitational oscillations. In this case we have $dR = 0$, $\sin \varphi = 0$, and $dR/\sin \varphi \rightarrow dz = - \alpha\, dp/g_z$, and we obtain for the frequency ν_g

$$\nu_g = \frac{1}{\alpha} \sqrt{R g_z A}. \tag{31}$$

If we take the streamline direction to be along the isobars, we obtain oscillations of purely inertial character. In this case $dp = 0$, and the frequency ν_i is

$$\nu_i = |\sin \varphi_p| \sqrt{\frac{R C}{g_z}}, \tag{32}$$

where φ_p is the angle in the meridional plane between the axis and the isobars. From eqs. (31) and (32) we see that the coefficient A may be regarded as a measure of the gravitational stability in the vortex, and C as a measure of the inertial (rotational) stability.

It follows from the Solberg — Høiland theory that the vortex may be unstable, even when A and C are both positive, if the baroclinity, measured by B, is sufficiently large. As shown by Høiland (1941), the stability criteria are obtained by demanding that ν^2 in eq. (30) is positive for all streamline directions, i. e. that the quadratic form within the brackets in eq. (30) is positive definite. This gives the stability criteria[2]

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[1] A list of the contributors to this theory is given by Holmboe (1948).

[2] Høiland (1941) defines stability in a slightly wider sense, including also the case when the quadratic form is zero for one direction, and positive for all other directions. The deviation from Høiland's admittedly correct definition is made for the sake of convenience.

$$\delta^2 > 0, \; A > 0, \; C > 0, \tag{33}$$

where

$$\delta^2 = AC - B^2 = \frac{\mu g_z^2}{R^5}\left(\frac{\partial \sigma}{\partial R}\frac{\partial c^2}{\partial p} - \frac{\partial c^2}{\partial R}\frac{\partial \sigma}{\partial p}\right). \tag{34}$$

The differential equation for the stream function (29) is seen to be of the elliptic, parabolic or hyperbolic type, according as $\delta^2 \gtreqless 0$. In particular, we obtain the result that the differential equation for the stream function (29) is of the elliptic type for all stable vortices[1].

If an unstable vortex were exposed to sources of heat or angular momentum, then unstable revolutions would in general be released, and the deviations from the state of balance would become appreciable. This contradicts the assumption on which the derivation of eq. (29) was based, namely, that the vortex is approximately in the state of balance at all times. Consequently, eq. (29) applies to stable vortices only[2]. We must therefore assume that the inequalities (33) are satisfied everywhere in the meridional plane. This implies that the differential equation (29) for the stream function is of the elliptic type, thus having the character of a generalized Poisson equation. Together with a suitable boundary condition, eq. (29) will determine the stream function uniquely.

5. The boundary condition.

If the fluid is enclosed within a rigid, thorus-shaped wall, which intersects the meridional plane along a closed curve S, then the boundary condition must express that the meridional motion on the boundary curve has no component normal to this curve. This may be expressed as a linear homogeneous equation in $\dot{R}$ and $D\Phi/dt$,

$$\frac{D\Phi}{dt} + b\dot{R} = \frac{\partial \Phi}{\partial t} + \dot{p}\frac{\partial \Phi}{\partial p} + \dot{R}\frac{\partial \Phi}{\partial R} + b\dot{R} = 0 \qquad \text{on } S,$$

say. Utilizing eqs. (5), (6), and (25), this can be written

[1] This statement cannot be reversed. In the case of an unstable vortex, the differential equation for the stream function may be elliptic, parabolic or hyperbolic.

[2] The possibility of a vortex with one indifferent direction, corresponding to a parabolic differential equation for the stream function, will not be considered here.

$$\frac{\partial \Phi}{\partial t} + \frac{a g_z}{R}\frac{\partial \psi}{\partial R} + \left(\frac{c^2 g_z}{R^4} + b\right)\frac{\partial \psi}{\partial p} = 0 \qquad \text{on } S. \tag{35}$$

The boundary condition would thus involve both of the unknown functions $\partial \Phi / \partial t$ and ψ. To be able to utilize this boundary condition, one would have to go back to the system (26), (27).

For simplicity, however, we shall here assume that our boundary curve S is a fixed, known curve in the coordinates R, p. Since S is a streamline in the Rp-plane, we know that ψ must be constant along S; this constant may without restriction be set equal to zero. The boundary condition is therefore

$$\psi = 0 \text{ on } S. \tag{36}$$

This boundary condition, together with differential equation (29), determines ψ uniquely inside S. The proof follows from Green's theorem, in the same way as in the case of the ordinary Poisson equation.

Our boundary value problem (29), (36) is also encountered in other branches of physics, e. g. equilibrium of a loaded, anisotropic membrane, and potential distribution in an anisotropic, conducting sheet with sources and sinks of electricity. The numerical solution of our boundary value problem by means of relaxation methods has been discussed by L. Tasny-Tschiassny (1949). No numerical solutions will be given in this paper; instead, we shall derive some general properties of the solution.

6. The Green function. Properties of the solution in the vicinity of a jump in *E* or *F*.

The solution of the problem (29), (36) may be written in the form

$$\psi(R,p) = \iint_{\gamma} G(R,p;R_0,p_0)\left[\left(\frac{\partial E}{\partial R}\right)_{R_0,\,p_0} + \left(\frac{\partial F}{\partial p}\right)_{R_0,\,p_0}\right] dR_0\, dp_0, \tag{37}$$

where γ is the region of the Rp-plane enclosed by the boundary curve S, and G is the Green function of the problem. We notice that if $\partial E/\partial R$ and $\partial F/\partial p$ vanish everywhere in γ, i. e. if E is constant along each isobar, and F constant along each line parallel to the axis, then the solution is $\psi \equiv 0$, and there will be no meridional motion in the Rp-plane. To set up meridional circulations, the sources of heat and angular momentum must be distributed in such a manner that E varies along the isobars, and F varies in the direction parallel to the axis.

The Green function represents the solution in the case of a sudden change (jump) of E along the isobars, or a sudden change of F along the lines parallel to the axis. More precisely, G represents the solution (apart from a constant factor) when $\partial E/\partial R$ and $\partial F/\partial p$ vanish everywhere except at the point R_0, p_0, where $\partial E/\partial R$ or $\partial F/\partial p$ is infinite. The Green function has a logarithmic singularity in the "jump" in E or F, i. e. in the point $R = R_0$, $p = p_0$. The behavior of G in the vicinity of this point is considered in some detail in the Appendix at the end of this paper. It is shown there that the principal part G_1 of G in the vicinity of R_0, p_0 is

$$(38)\quad G_1 = \frac{1}{2\pi\delta_0} \ln\left[C_0(R-R_0)^2 - 2B_0(R-R_0)(p-p_0) + A_0(p-p_0)^2\right]^{\frac{1}{2}},$$

where the subscript "0" denotes values at the point R_0, p_0. Apart from a constant factor, this formula represents the solution in the vicinity of a jump in E or F.

The formula (38) may be regarded as a generalization of the formula for two-dimensional, irrotational motion around a rectilinear vortex filament, or for the magnetic field around a linear electric current. The only difference is that the field represented by (38) is distorted as a result of different stability conditions in different directions, so that we obtain elliptical streamlines instead of circular ones. The elliptical streamlines (G_1 = constant) in the Rp-plane are seen to be conformal and concentric, with the jump (R_0, p_0) in the center. Since we are considering the field in the vicinity of the jump only, we may regard p as a linear function og R and z. The streamlines are therefore conformal and concentric ellipses also when represented in the Rz-plane (Fig. 3).

If the meridional motion is produced by a jump in E along an isobar, the sense of the circulation along the elliptical streamlines is seen to be such that the particles are moving toward lower pressure ($\dot{p}$ negative) on the side of the jump where E is larger, and toward higher pressure ($\dot{p}$ positive) on the other side of the jump, where E is smaller. If the motion is produced by a jump in F along a line

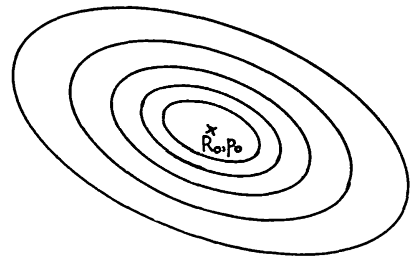

Fig. 3. Meridional streamlines in the vicinity of a jump in E or F.

parallel to the axis, then the particles will move away from the axis ($\dot{R}$ positive) on the side of the jump where F is larger, and toward the axis ($\dot{R}$ negative) on the other side of the jump, where F is smaller.

Since δ may be regarded as a measure of the stability of the vortex, it follows from (38) that for a given jump in E or F, the meridional circulation will be the stronger, the weaker the stability of the vortex.

In order to determine the shape and orientation of the elliptical streamlines, we notice that $\partial G_1/\partial p$ vanishes along the line

$$\frac{p - p_0}{R - R_0} = \frac{B_0}{A_0} = -\frac{(\partial \sigma/\partial R)_0}{(\partial \sigma/\partial p)_0}, \tag{39}$$

i. e. along the isentropic line $\sigma = \sigma_0$ through the center. This means that the isentropic line through the center will intersect the streamlines in points where the streamlines run parallel to the axis of the vortex (Fig. 4). The lines $\sigma = \sigma_0$ and $R = R_0$ are therefore conjugate diameters in each elliptical streamline. The shape of the ellipses is known if we know the ratio between the lengths of these diameters. Instead of the

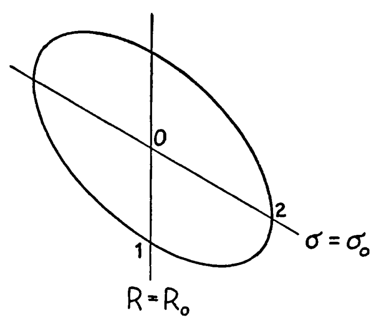

Fig. 4. The conjugate diameters $R = R_0$ and $\sigma = \sigma_0$.

true lengths, we may as well use increments in R or p. Denoting by 1 the point of intersection between one elliptical streamline and the line $R = R_0$, and by 2 the point of intersection between the streamline and the isentropic line $\sigma = \sigma_0$ (Fig. 4), we find

$$\left|\frac{p_1 - p_0}{R_2 - R_0}\right| = \frac{\delta_0}{A_0} = \sqrt{\frac{B_0}{A_0}\left(\frac{C_0}{B_0} - \frac{B_0}{A_0}\right)} = \sqrt{\left(\frac{dp}{dR}\right)_\sigma \left[\left(\frac{dp}{dR}\right)_c - \left(\frac{dp}{dR}\right)_\sigma\right]}.$$

Here $(dp/dR)_\sigma$ and $(dp/dR)_c$ denote the slopes in the Rp-plane of the lines $\sigma =$ const. and the lines $c =$ const., respectively. From this formula, the streamlines may be constructed.

Similarly, we find that $\partial G_1/\partial R$ vanishes along the line

$$\frac{p - p_0}{R - R_0} = \frac{C_0}{B_0} = -\frac{(\partial c^2/\partial R)_0}{(\partial c^2/\partial p)_0}.$$

This shows that the line $c = c_0$ through the center intersects the elliptical streamlines in points where the streamlines touch the isobars (Fig. 5). The lines $c = c_0$ and $p = p_0$ are therefore conjugate diameters in each elliptical streamline. Denoting by 3 the point of intersection between a streamline and the line $c = c_0$, and by 4 the point of intersection between the same streamline and the isobar $p = p_0$ (Fig. 5), we find

$$\left|\frac{p_3 - p_0}{R_4 - R_0}\right| = \frac{C_0}{\delta_0} = \left[\frac{B_0}{C_0}\left(\frac{A_0}{B_0} - \frac{B_0}{C_0}\right)\right]^{-\frac{1}{2}} = \left\{\left(\frac{dR}{dp}\right)_c \left[\left(\frac{dR}{dp}\right)_\sigma - \left(\frac{dR}{dp}\right)_c\right]\right\}^{-\frac{1}{2}}.$$

This formula determines the shape of the streamlines, in the same way as does eq. (40).

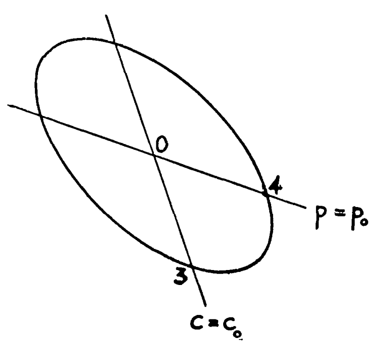

Fig. 5. The conjugate diameters $c = c_0$ and $p = p_0$.

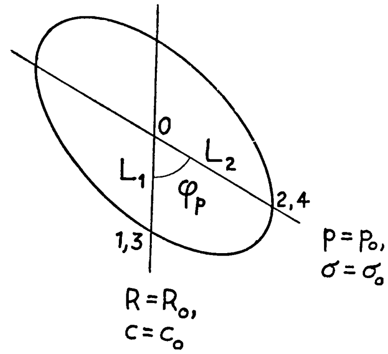

Fig. 6. Conjugate diameters of an elliptical streamline in a barotropic vortex.

As an illustration we consider the case of a barotropic vortex ($B = 0$, $\delta^2 = A\,C$). The isentropic lines will then coincide with the isobars, and from eq. (10) we see that the lines c = constant will coincide with the lines R = constant (Fig. 6). The points 1 and 2 in Fig. 4 will coincide with the points 3 and 4 in Fig. 5, respectively. Eqs. (40) and (42) become identical and assume the form

$$\left|\frac{p_1 - p_0}{R_2 - R_0}\right| = \sqrt{\frac{C_0}{A_0}}. \tag{43}$$

We introduce the true lengths $2\,L_1$ and $2\,L_2$ of the conjugate diameters $R = R_0$ and $p = p_0$, and find

$$L_1 = \frac{\alpha}{g_z}\,|\,p_1 - p_0\,|, \quad L_2 = \left|\frac{R_2 - R_0}{\sin\,\varphi_p}\right| \tag{44}$$

where φ_p is the angle between the isobar direction and the axis. Using eqs. (31), (32), and (44), we may write (43) in the form

$$\frac{L_1}{L_2} = \frac{\nu_i}{\nu_g}. \tag{45}$$

It follows from this formula that if the gravitational stability (ν_g) is weak compared to the inertial stability (ν_i), then the streamlines are long and narrow ellipses with their major axis nearly parallel to the axis of the vortex (Fig. 7 a). On the other hand, if the inertial stability is weak compared to the gravitational stability, then the streamlines will be long and narrow ellipses with their major axis nearly parallel to the isobars (Fig. 7 b). In the atmosphere, the ratio ν_i/ν_g is normally

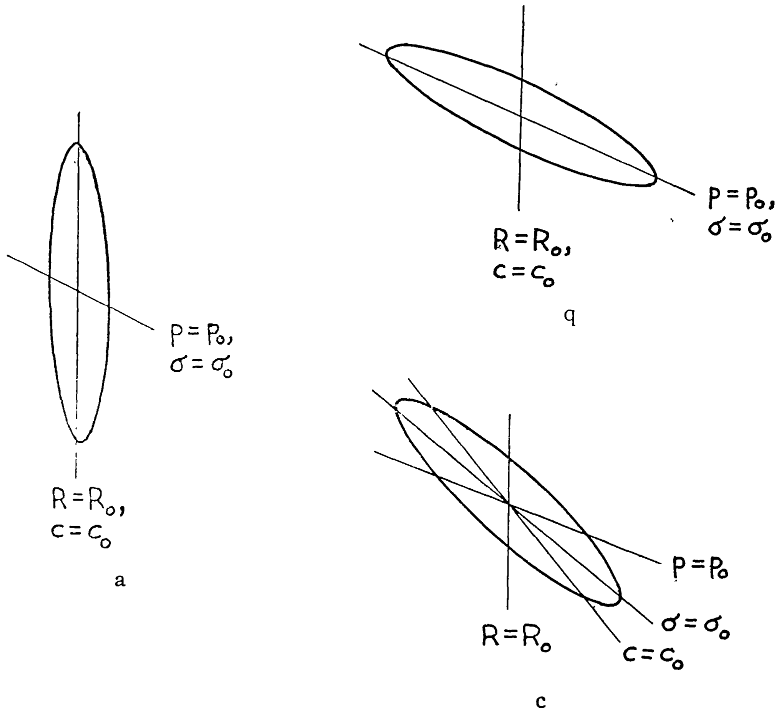

Fig. 7. Shape and orientation of elliptical streamline.
a. Barotropic vortex with strong inertial stability.
b. Barotropic vortex with strong gravitational stability.
c. Extremely baroclinic vortex.

of the order 1/100. Thus, if part of the atmosphere is regarded as a barotropic vortex, the streamlines in the vicinity of a jump in E or F would be of the latter type, shown in Fig. 7 b.

As a third example we consider the case of strong baroclinity, where δ is small because the lines $c =$ constant form a small angle with the lines $\sigma =$ constant (Fig. 7 c). More presicely, we will assume that

$$\delta \ll \frac{a\, C}{g_z}, \quad \delta \ll \frac{g_z\, A}{a}. \tag{46}$$

Then it follows from eqs. (40) and (42) that the streamlines are long and narrow ellipses with their major axes in a direction close to the directions of the lines $c =$ constant and $\sigma =$ constant (Fig. 7 c).

In all the examples considered we have seen that meridional motion takes place mainly in the direction of weakest stability, a result the correctness of which will be felt intuitively.

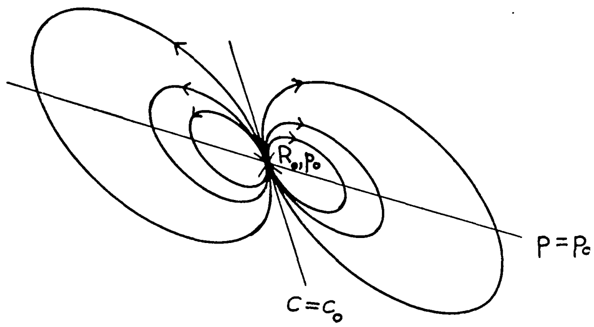

Fig. 8. Meridional streamlines in the vicinity of a point-source of heat.

7. Meridional circulation in the vicinity of a point-source of heat.

The meridional circulation produced by heat sources alone is obtained from eq. (37) by setting $F = 0$. Integrating by parts, we obtain the solution in the form

$$\psi\,(R, p) = -\iint\limits_{\gamma} \frac{\partial G}{\partial R_0} E(R_0, p_0)\, d R_0\, d p_0 \tag{47}$$

since, by definition

$$G = 0 \quad \text{on } S. \tag{48}$$

Hence we see that $-\partial G/\partial R_0$ represents the solution in the case of a single point-source of heat of unit strength[1], located at (R_0, p_0). The principal part of $-\partial G/\partial R_0$ in the singular point (R_0, p_0) is $-\partial G_1/\partial R_0$. According to (38), we have

$$-\frac{\partial G_1}{\partial R_0} = \frac{1}{2\pi\delta_0}\,\frac{C_0(R-R_0) - B_0(p-p_0)}{C_0(R-R_0)^2 - 2B_0(R-R_0)(p-p_0) + A(p-p_0)^2}. \tag{49}$$

This is therefore the stream function in the vicinity of a point-source of heat of unit strength.

The streamlines ($-\partial G_1/\partial R_0 =$ constant) are conformal ellipses of the same shape and orientation as those considered in section 6; they may therefore be constructed by means of eqs. (40) or (42). However, the streamlines are in the present case not concentric, but are displaced so that they all run through the point-source, whereas

[1] In the meaning $\iint\limits_{\gamma} E\, dR\, dp = 1$. The word "point-source" refers to the meridional plane. In space, the heat source considered is located along a circle around the axis.

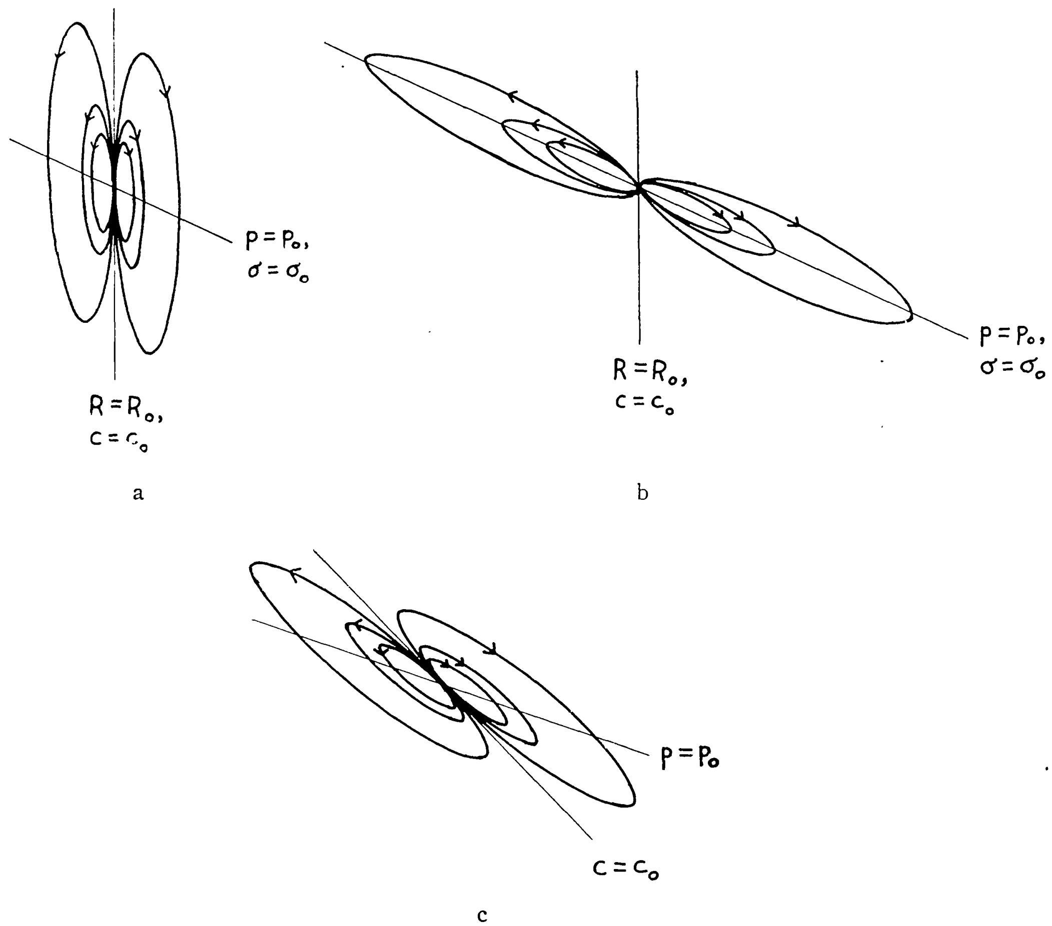

Fig. 9. Meridional streamlines in the vicinity of a point-source of heat.
a. Barotropic vortex with strong inertial stability.
b. Barotropic vortex with strong gravitational stability.
c. Extremely baroclinic vortex.

their centers are located on the isobar $p = p_0$ running through the point-source (Fig. 8). It was shown in section 6 that the isobar and the line $c =$ constant through the center of an elliptic streamline are conjugate diameters. Hence it follows that all streamlines will touch the line $c = c_0$ through the point source. The field is a distorted field of a dipole.

The sense of the circulation is such that the motion is directed toward lower pressure ($\dot{p}$ negative) through a heat source, and toward higher pressure ($\dot{p}$ positive) through a cold source. Again we see, as

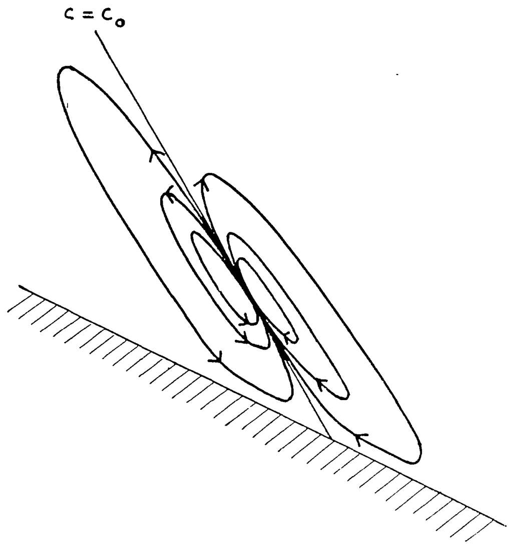

Fig. 10. Streamlines of meridional motion produced by a heat source near a horizontal boundary in a strongly baroclinic vortex.

in the previous section, that for a given heat source, the meridional motion will be the stronger, the weaker the stability of the vortex (expressed by δ).

In the three special cases considered in section 6, we obtain streamline patterns as shown in Fig. 9. Fig. 9 a shows the streamlines in the vicinity of a point-source of heat in a barotropic vortex with weak gravitational stability compared to the inertial stability. Fig. 9 b shows the case of a barotropic vortex with weak inertial stability compared to the gravitational stability. This case corresponds to normal atmospheric conditions in regions where the baroclinity is weak. Fig. 9 c shows the case of strong baroclinity. This again illustrates that the motion takes place mainly in the direction where the resistance of the stabilizing forces is weakest, as was mentioned in section 6.

As allready stressed, the streamline patterns discussed apply to the close vicinity of the point-source only. As we go further away from the point-source, the streamlines will change their form so as to adjust themselves to the shape of the boundary. Thus, if the point-source of heat is situated near a horizontal boundary in a vortex with strong baroclinity, one should expect a streamline pattern as shown in Fig. 10. This picture may be regarded as an illustration of the streamline pattern in a plane normal to a frontal surface, the "point-source" of heat being represented by the released heat of condensation in the frontal cloud. One should therefore expect the frontal surface to slope roughly along the isentropic surfaces, in agreement with observation.

8. Meridional circulation in the vicinity of a point-source of angular momentum.

The meridional motion produced by sources of angular momentum alone is arrived at by putting $E = 0$ in the solution (37). Integrating by parts, we obtain on account of (48),

$$\psi(R, p) = -\iint_{\gamma} \frac{\partial G}{\partial p_0} F(R_0, p_0)\, d R_0\, d p_0 . \tag{50}$$

In the case of a single point-source of angular momentum of unit strength[1], situated at the point R_0, p_0, the solution is $\psi = -\partial G/\partial p_0$. In the vicinity of the point-source, this function may be approximated by its principal part, which is, according to (38),

$$-\frac{\partial G_1}{\partial p_0} = \frac{1}{2\pi\delta_0} \frac{-B_0(R-R_0) + A_0(p-p_0)}{C_0(R-R_0)^2 - 2B_0(R-R_0)(p-p_0) + A_0(p-p_0)^2}.$$

The streamlines ($-\partial G_1/\partial p_0 =$ constant) are conformal ellipses of the same shape and orientation as in the cases considered in sections 6 and 7. All streamlines will run through the point-source of angular momentum, and will have their centers situated on the line $R = R_0$ through the point-source (Fig. 11). Since the lines $R =$ constant and $\sigma =$ constant represent the directions of a pair of conjugate diameters in each elliptical streamline (section 6), it follows that all streamlines will in the point-source touch the isentropic line $\sigma = \sigma_0$.

The sense of the circulation along these streamlines is seen to be such that the motion through a source of angular momentum is directed away from the axis of the vortex, and the motion through a sink of angular momentum is directed toward the axis of the vortex. Again we see that the meridional circulation will be the stronger, the

[1] In the meaning $\iint_{\gamma} F\, d R\, d p = 1$.

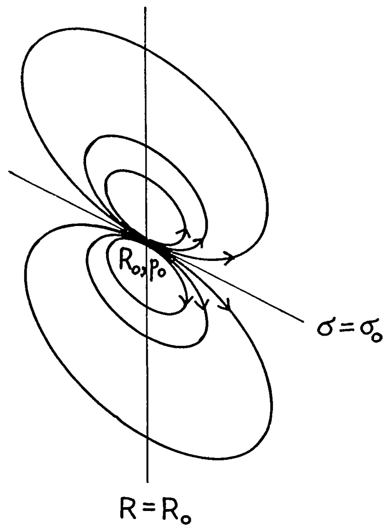

Fig. 11. Meridional streamlines in the vicinity of a point-source of angular momentum.

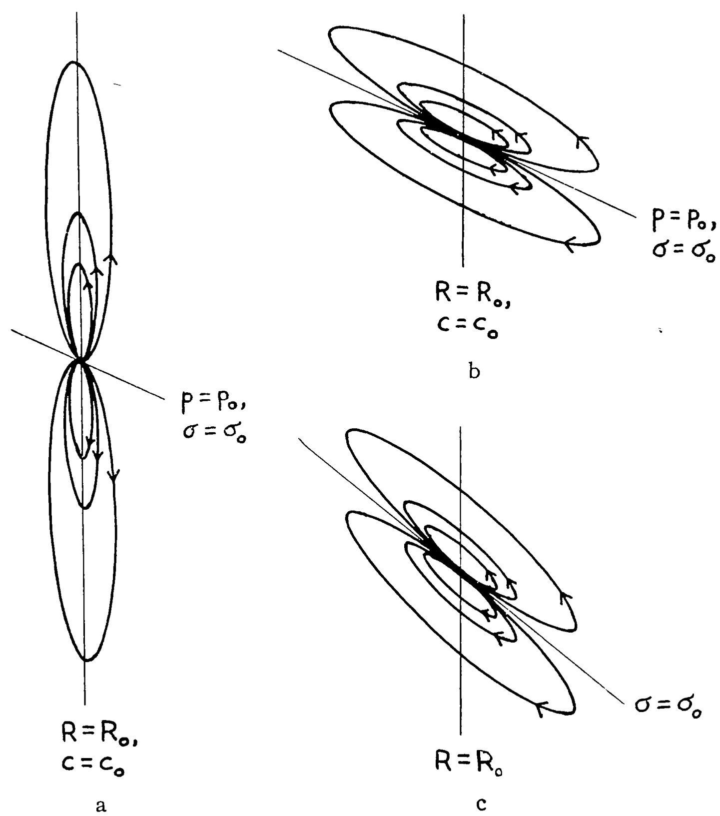

Fig. 12. Meridional streamlines in a vicinity of a point-source of angular momentum.

a. Barotropic vortex with strong inertial stability.
b. Barotropic vortex with strong gravitational stability.
c. Extremely baroclinic vortex.

weaker the stability of the vortex, the source of angular momentum being the same.

The meridional circulation in the vicinity of a point-source of angular momentum in a barotropic vortex with weak gravitational stability compared to the inertial stability is shown in Fig. 12 a. Fig. 12 b shows the case of a barotropic vortex with weak inertial stability compared to the gravitational stability, and Fig. 12 c shows the case of a vortex with strong baroclinity.

9. Meridional circulations driven by friction at the boundary.

The field around a point-source of angular momentum, considered in the preceding section, is invalid near the boundary, where the stream-lines must fit in with the boundary condition. It is possible, however, to investigate the field around a source of angular momentum at the boundary. This case is especially important for applications to the atmosphere, because the strongest frictional forces are found in the layer near the surface of the earth.

In Fig. 13, the line SS is part of the boundary, and P is a point close to the boundary. We assume that along the line $R =$ constant through P, $F = 0$ above P, whereas $F = F_1 =$ constant below P; in all other points, $F = 0$. There is then a jump in F, or a point-source in $\partial F/\partial p$ at the point P. According to (37), the corresponding solution is

$$\psi = K\,G(R, p;\ R_0, p_0)$$

where K is a constant, R_0 and p_0 are the coordinates of the point P. Disregarding the boundary, the field in the vicinity of P may be approximated by

$$\psi = KG_1(R, p;\, R_0, p_0).$$

However, this field violates the boundary condition (36). We may fulfil the boundary condition by adding the field of a similar point-source of opposite sign, situated at the point P' *outside* the boundary

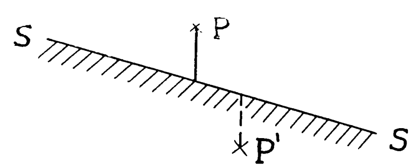

Fig. 13. Method of image.

(method of image). When the points P and P' are infinitely close together, and the coordinates of the point P' are $R_0 - dR_0$, $p_0 - dp_0$, the resulting field is

$$\psi = K\left(dR_0 \frac{\partial G_1}{\partial R_0} + dp_0 \frac{\partial G_1}{\partial p_0}\right). \tag{52}$$

Inserting here the expression (38) for G_1, we obtain

$$\psi = K_1 \frac{p - p_0 - \dfrac{C_0 - B_0\, dp_0/dR_0}{B_0 - A_0\, dp_0/dR_0}(R - R_0)}{C_0 (R - R_0)^2 - 2 B_0 (R - R_0)(p - p_0) + A_0 (p - p_0)^2} \tag{53}$$

where K_1 is another constant. The streamline $\psi = 0$ is the straight line

$$p - p_0 = \frac{C_0 - B_0\, dp_0/dR_0}{B_0 - A_0\, dp_0/dR_0}(R - R_0). \tag{54}$$

The boundary condition (36) is fulfilled when this rectilinear streamline coincides with the boundary. If the equation of the tangent of the boundary is

$$p - p_0 = \vartheta (R - R_0), \tag{55}$$

then we must choose dp_0/dR_0 such that

$$\frac{C_0 - B_0\, dp_0/dR_0}{B_0 - A_0\, dp_0/dR_0} = \vartheta. \tag{56}$$

Eq. (53) then becomes

$$\psi = K_1 \frac{p - p_0 - \vartheta (R - R_0)}{C_0 (R - R_0)^2 - 2 B_0 (R - R_0)(p - p_0) + A_0 (p - p_0)^2}. \tag{57}$$

This is the solution in the vicinity of a point-source of angular momentum at the boundary. The streamlines ψ = constant are ellipses of the same shape and orientation as in the cases previously considered. All streamlines run through the point-source, where they touch the boundary (Fig. 14). It is easy to see how this picture is modified in the three special cases considered in the preceding sections.

In the atmosphere, surface friction will represent a source of angular momentum in zones of easterly wind at the surface of the earth, and a sink of angular momentum in zones of surface westerlies. The effect of surface friction within the zone of strong surface

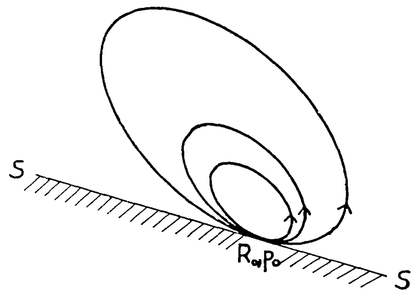

Fig. 14. Meridional streamlines in the vicinity of a point-source of angular momentum at the boundary.

westerlies in middle latitudes may be represented by a number of point-sinks of angular momentum. Thus we see that the meridional circulation set up by surface friction in the zone of westerlies in the case of strong baroclinity must be qualitatively as shown in Fig. 15. This picture fits in well with the high frequency of cloudiness and precipitation observed just north of the zone of surface westerlies. However, to this meridional circulation must be added the circulations caused by heat sources and by friction above the surface layer.

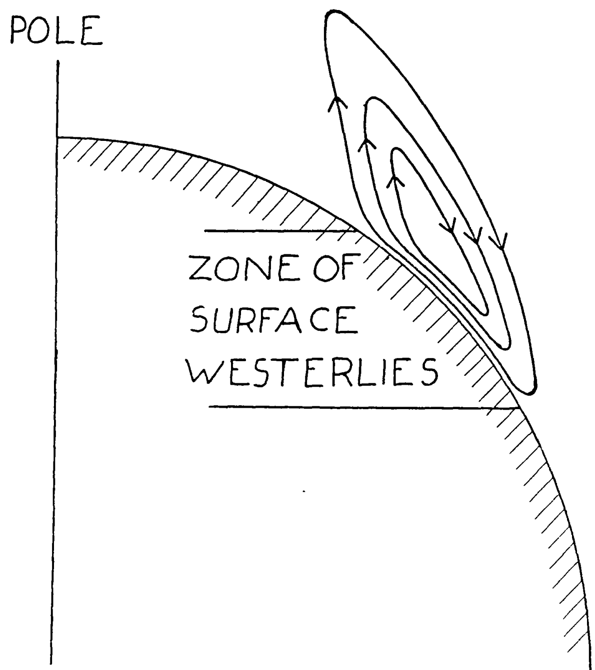

Fig. 15. Meridional circulation in the atmosphere, produced by surface friction in the baroclinic zone of surface westerlies.

10. Energy considerations.

We have assumed the meridional currents to be very slow compared to the vortex motion, so that the direct effect of the meridional circulations on the field of motion is slight. We find this assumption justified in the case of the atmosphere, where the organized, non-geostrophic meridional circulation is so weak that it is barely detectible from the wind data. The importance of such weak meridional currents, however, does not lie in their direct effect on the wind field, but on their ability to change profoundly the structure of the vortex by transporting the fields of the quasi-conservative properties c and σ. Some insight into this mechanism may be gained by examining the changes of enthalpy and kinetic energy within the vortex.

We shall assume that heat sources and friction are operating simultaneously. It follows from the linearity of our problem that the resulting circulation is then obtained simply by adding the motions caused by each separate effect.

Denoting by h enthalpy per unit mass, and by dm the mass element, the total enthalpy of the system is

$$H = \int h\, dm, \tag{58}$$

where the integral is taken over the entire system. Differentiating with respect to time, and applying a well-known equation of thermodynamics, viz.

$$\frac{Dh}{dt} = T\frac{D\sigma}{dt} + \alpha\frac{Dp}{dt} \tag{59}$$

we obtain, since $\frac{D}{dt}\, dm = 0$, and since there is no mass transport across the boundary curve S,

$$\frac{dH}{dt} = \int T\frac{D\sigma}{dt}\, dm + \int \alpha \dot{p}\, dm. \tag{60}$$

Here, the first term on the right represents the total heat per unit time given to the system from external sources, plus the frictional dissipation within the system. The second term on the right represents the increase of enthalpy caused by the meridional currents. On account of eq. (25), this term may be written

$$\int \alpha \dot{p}\, dm = -\int \alpha \frac{g_z}{R} \frac{\partial \psi}{\partial R} dm.$$

Choosing here as mass elements the ring-shaped elements considered in section 3, we may write

$$dm = 2\pi \frac{R}{g_z} dR\, dp \tag{61}$$

in agreement with (21). Thus we obtain, on integrating by parts, and utilizing eqs. (7) and (28),

$$\begin{aligned} \int \alpha \dot{p}\, dm &= -2\pi \iint \alpha \frac{\partial \psi}{\partial R} dR\, dp = 2\pi \iint \frac{\partial \alpha}{\partial R} \psi\, dR\, dp \\ &= -2\pi \iint \frac{1}{R^3} \frac{\partial c^2}{\partial p} \psi\, dR\, dp = 2\pi \iint \frac{R}{g_z} B \psi\, dR\, dp = \int B \psi\, dm. \end{aligned} \tag{62}$$

The rate of increase of enthalpy is therefore

$$\frac{dH}{dt} = \int T \frac{D\sigma}{dt} dm + \int B \psi\, dm. \tag{63}$$

The kinetic energy of the vortex motion is $c^2/2R^2$ per unit mass, or

$$K = \int \frac{c^2}{2R^2} dm \tag{64}$$

for the entire system. In virtue of (1), we have

$$\frac{D}{dt} \frac{c^2}{2R^2} = \frac{1}{2R^2} \frac{Dc^2}{dt} - \frac{c^2}{R^3} \dot{R} = \omega \frac{Dc}{dt} - \frac{c^2}{R^3} \dot{R}, \tag{65}$$

so that the rate of change of the total kinetic energy is

$$\frac{dK}{dt} = \int \omega \frac{Dc}{dt} dm - \int \frac{c^2}{R^3} \dot{R}\, dm. \tag{66}$$

In this equation, the first term on the right represents the work per unit time done by the (frictional) forces that change the angular momentum; and the second term represents the effect of the meridional motion. Using eqs. (25), (61), and (28), the second term on the right may be written

$$(67)\quad \begin{aligned} -\int \frac{c^2}{R^3}\dot{R}\,dm &= -2\pi\iint \frac{c^2}{R^3}\frac{\partial\psi}{\partial p}\,dR\,dp = 2\pi\iint \frac{1}{R^3}\frac{\partial c^2}{d\,p}\psi\,dR\,dp \\ &= -2\pi\iint \frac{R}{g_z}B\,\psi\,dR\,dp = -\int B\,\psi\,dm. \end{aligned}$$

The rate of increase of kinetic energy is therefore

$$(68)\qquad \frac{dK}{dt} = \int \omega\,\frac{Dc}{dt}\,dm - \int B\,\psi\,dm.$$

Inspection of (63) and (68) shows that the term $\int B\psi\,dm$ represents a transformation of enthalpy into kinetic energy or vice versa. If $\int B\psi\,dm < 0$, enthalpy is transformed into kinetic energy, and we have what is usually called a "direct" circulation. If $\int B\psi\,dm > 0$, kinetic energy is transformed into enthalpy, and the circulation is "indirect".

The change of kinetic energy caused by heat sources is entirely due to the second term on the right of (68). The heat sources will therefore give an increase, or a decrease in kinetic energy, according as the meridional circulation set up by the heat sources is a direct or an indirect one.

Friction, on the other hand, will act in two different ways to change the kinetic energy. In the first place, the work done by the frictional force will give a change in kinetic energy, expressed by the first term on the right in (68). In the second place, friction will set up meridional circulations which lead to a transformation of enthalpy into kinetic energy or vice verca. This effect corresponds to the last term on the right of (68). These two effects may counteract or cooperate, depending on whether the frictionally driven circulation is direct or indirect. The total effect of friction on the kinetic energy thus depends very much on the accompanying meridional circulation.

It is possible to express the "energy transformation" $-\int B\psi\,dm$ directly through E and F. For this purpose we introduce a function $\Psi(R, p)$, defined by

$$(69)\quad L(\Psi) = \frac{\partial}{\partial R}\left(A\frac{\partial\Psi}{\partial R} + B\frac{\partial\Psi}{\partial p}\right) + \frac{\partial}{d\,p}\left(B\frac{\partial\Psi}{\partial R} + C\frac{\partial\Psi}{\partial p}\right) = \frac{R}{g_z}B,$$

$$(70)\qquad \Psi = 0 \quad \text{on } S.$$

It will be seen that Ψ is uniquely determined when the structure of the vortex is known. Thus Ψ does not depend on the heat sources (E) and the frictional forces (F) in operation.

Since Ψ has the dimension of [$\psi \times$ time], we may define a meridional displacement field ($\triangle R$, $\triangle p$), with the lines $\Psi =$ constant as streamlines, by

$$\triangle R = \frac{g_z}{R}\frac{\partial \Psi}{\partial p}, \qquad \triangle p = -\frac{g_z}{R}\frac{\partial \Psi}{\partial R}. \tag{71}$$

Comparison with eq. (25) shows that this displacement field will satisfy the equation of continuity. It follows from what was said in section 6 that the lines $\Psi =$ constant are closed curves, encircling regions of strong baroclinity, and that the displacement field (71) circulates in a positive sense around the solenoids in the meridional plane.

From (62) and (69),

$$-\int B\psi\, dm = -2\pi \iint \frac{R}{g_z} B\psi\, dR\, dp = -2\pi \iint L(\Psi)\psi\, dR\, dp.$$

Using Green's theorem we have, since L (Ψ) is self-adjoint,

$$\iint L(\Psi)\psi\, dR\, dp = \iint L(\psi)\, \Psi\, dR\, dp.$$

Utilizing further eqs. (29), (61), (70), and (71), we finally obtain

$$-\int B\psi\, dm = -2\pi \iint L(\psi)\, \Psi\, dR\, dp$$

$$= -2\pi \iint \left(\frac{\partial E}{\partial R} + \frac{\partial F}{\partial p}\right)\Psi\, dR\, dp = 2\pi \iint \left(E\frac{\partial \Psi}{\partial R} + F\frac{\partial \Psi}{\partial p}\right) dR\, dp \tag{72}$$

$$= 2\pi \iint \left(-E\triangle p + F\triangle R\right)\frac{R}{g_z}\, dR\, dp = \int \left(-E\triangle p + F\triangle R\right) dm.$$

Thus it is not necessary first to determine ψ in order to calculate the energy transformation. Having determined Ψ, it is an easy matter to calculate from (72) the contribution to the energy transformation of any sources of heat or angular momentum. In particular, we can decide immediately whether a certain point source of heat or angular momentum will set up a direct or an indirect circulation. In the case of a baroclinic vortex with $B > 0$ we thus infer, since the displacement field ($\triangle R$, $\triangle p$) circulates around the solenoids in a positive sense, that heat sources in the outer, warmer part of the vortex, and heat sinks in the inner, colder part will give direct circulations. Likewise, sources of angular momentum in the lower part of the vortex, where the vortex motion is slow, and sinks of angular momentum in the upper part, where the vortex motion is fast, will give direct meridional circulations. Thus sinks ot angular momentum in the lower part of the vortex will be more effective in reducing the kinetic energy of the vortex than will be sinks of angular momentum in the upper part.

11. Change in the structure of the vortex. The stationary vortex.

Eliminating $\partial \Phi / \partial t$ between eqs. (11) and (26) and between (12) and (27), we find

$$\mu \frac{\partial \sigma}{\partial t} = E - A \frac{\partial \psi}{\partial R} - B \frac{\partial \psi}{\partial p} \tag{73}$$

$$\frac{1}{R^3} \frac{\partial c^2}{\partial t} = F - B \frac{\partial \psi}{\partial R} - C \frac{\partial \psi}{\partial p}. \tag{74}$$

Having determined ψ, we may use these equations to calculate the rate of change of σ and c.

Hence it follows that for a stationary state, where σ and c do not change with time, we must have

$$E = A \frac{\partial \psi}{\partial R} + B \frac{\partial \psi}{\partial p} \tag{75}$$

$$F = B \frac{\partial \psi}{\partial R} + C \frac{\partial \psi}{\partial p}. \tag{76}$$

Eliminating ψ between these equations, we obtain

$$\frac{\partial}{\partial R} \frac{BE - AF}{AC - B^2} + \frac{\partial}{\partial p} \frac{CE - BF}{AC - B^2} = 0. \tag{77}$$

Furthermore, the stationary meridional circulation must fulfil the boundary condition (36), which may be written

$$-\frac{\frac{\partial \psi}{\partial R}}{\frac{\partial \psi}{\partial p}} = \left(\frac{dp}{dR}\right)_S \qquad \text{on } S,$$

where the right hand side denotes the slope of the boundary curve in the Rp-plane.

Eliminating $\partial \psi / \partial R$ and $\partial \psi / \partial p$ between this equation and (75) and (76), we find

$$\text{(78)} \qquad \frac{CE - BF}{BE - AF} = \left(\frac{dp}{dR}\right)_S \qquad \text{on } S.$$

The conditions (77) and (78) must be satisfied by the functions A, B, C, E, and F in any stationary state. If we have free disposal of the sources of heat and angular momentum, then any meridional circulation in any stable vortex may clearly be made stationary by assigning these sources so that (75) and (76), and hence (77) and (78) are fulfilled. However, in actual vortices such as the atmosphere, the radiative and turbulent heat flux, and the turbulent frictional stress depends on the fields of motion and temperature within the vortex. Thus E and F may be considered as certain (poorely known) functions of σ and c and their spatial derivatives. Eqs. (77) and (78) will then represent relationships between the fields of σ and c. The existence of stationary states depends on whether there are fields of σ and c that satisfy eqs. (77) and (78), as well as (10), and also on whether these fields correspond to stable vortices.

In a vortex which is not in a stationary state, the fields of σ and c, and hence the sources of heat and angular momentum and the meridional circulation, must continually change with time. Very little can be said about the characteristics of such changes. If a stationary state exists, one would perhaps expect that the vortex would approach this stationary state asymptotically. If there are no stationary states, then the changes must go on for ever.

12. Application to the atmosphere.

Axially symmetric meridional circulations, of the type discussed above, play a dominant role in the "old-fashioned" theories of the general circulation of the atmosphere. More recent ideas of the general circulation, on the other hand, tend to stress the importance of such motions which are asymmetric with respect to the earth's axis, e. g. wave disturbances, in producing the transfer of angular momentum necessary to maintain the wind currents, and also in maintaining their kinetic energy. The question of the relative importance of meridional circulations and asymmetric motions is far from being settled; and the following remarks on the character of possible meridional circulations in the atmosphere and their possible role in maintaining the wind fields are therefore presumably not yet entirely without interest.

In an attempt to apply the theory of meridional circulations to atmospheric motion, we are faced with the difficulty that the atmosphere is not a symmetric vortex around the earth's axis, since the fields ot motion and state vary from one meridional plane to another, whereas the theory has been developed for a strictly symmetric vortex. One way to overcome this difficulty would be to apply the theory of meridional circulations to the symmetric vortex formed by averaging over all longitudes. Then all asymmetric motions would enter into the theory as eddy motion, causing heat transfer and frictional forces of a quite complicated nature. The corresponding eddy transport of angular momentum is identical with the mechanism for transport of angular momentum suggested by Jeffreys (1926).

A somewhat different method seems plausible if, as is often observed, the asymmetric motion consists in smooth long waves in the westerlies around the globe, such that the state and the speed of the particles are fairly constant along the streamlines of the relative motion. Then it seems reasonable to regard the meridional circulation as being superimposed upon the wave motion, i. e. to assume that the meridional circulation will take place as if the wave motion were absent. To obtain the meridional circulation in pure form, we must therefore straighten out the waves by displacing the air poleward in the wave throughs and equatorward in the wave crests, thus forming a fictitious, approximately symmetric vortex. The fields of state and zonal motion in any meridional cross-section of this fictitious vortex will, in a first approximation, be represented by the fields of state and zonal motion in one typical meridional cross-section through the atmosphere. Our fictitious symmetric vortex will therefore have all the typical features of meridional cross-section through the atmosphere, whereas these typical features will be smoothed out if a symmetric vortex is formed by averaging over all longitudes.

As an example, a cross-section taken from a paper by Palmén and Nagler (1948) has been chosen. This cross-section represents a mean of several cross-sections through North America for 0300 GCT, 30 November 1946, but the averaging is done in such a way that the typical features are not smoothed out. The cross-section is shown in Fig. 16. The isentropic lines (dashed) and the lines of constant zonal wind speed (dotted) have been copied from the paper by Palmén and Nagler. The solid lines represent the lines of constant absolute angular momentum (c) in the corresponding fictitious, symmetric vortex. These lines have been constructed by means of the formula

$$\frac{c}{a} = \Omega a \cos^2 \varphi + u \cos \varphi, \tag{79}$$

where a means radius of the earth, Ω angular velocity of the earth's rotation, u relative zonal wind speed, and φ latitude. Lines are drawn for $c/a =$ a multiple of 10 m sec^{-1}. The coordinates of the diagram are latitude and pressure. The scale is such that vertical lengths are enlarged about 160 times compared to horizontal lengths. Lines parallel to the earth's axis, i. e. the lines $R =$ constant, are therefore approximately vertical lines in the diagram.

North of 50° latitude, the lines $c =$ constant are nearly coinciding with the vertical lines $R =$ constant, and the isentropic lines are nearly coinciding with the isobars. This means that the vortex in this region is almost barotropic. South of 50° latitude, on the other hand, the baroclinity is conspicuous, especially in the zone between 40° and 47°. Instability with respect to axially symmetric oscillations is found only in a comparatively small region (hatched), just south of the west wind maximum, where the slope of the isentropic lines is seen to be steeper than the slope of the lines $c =$ constant. Everywhere else in the cross-section, the slope of the lines $c =$ constant is seen to be

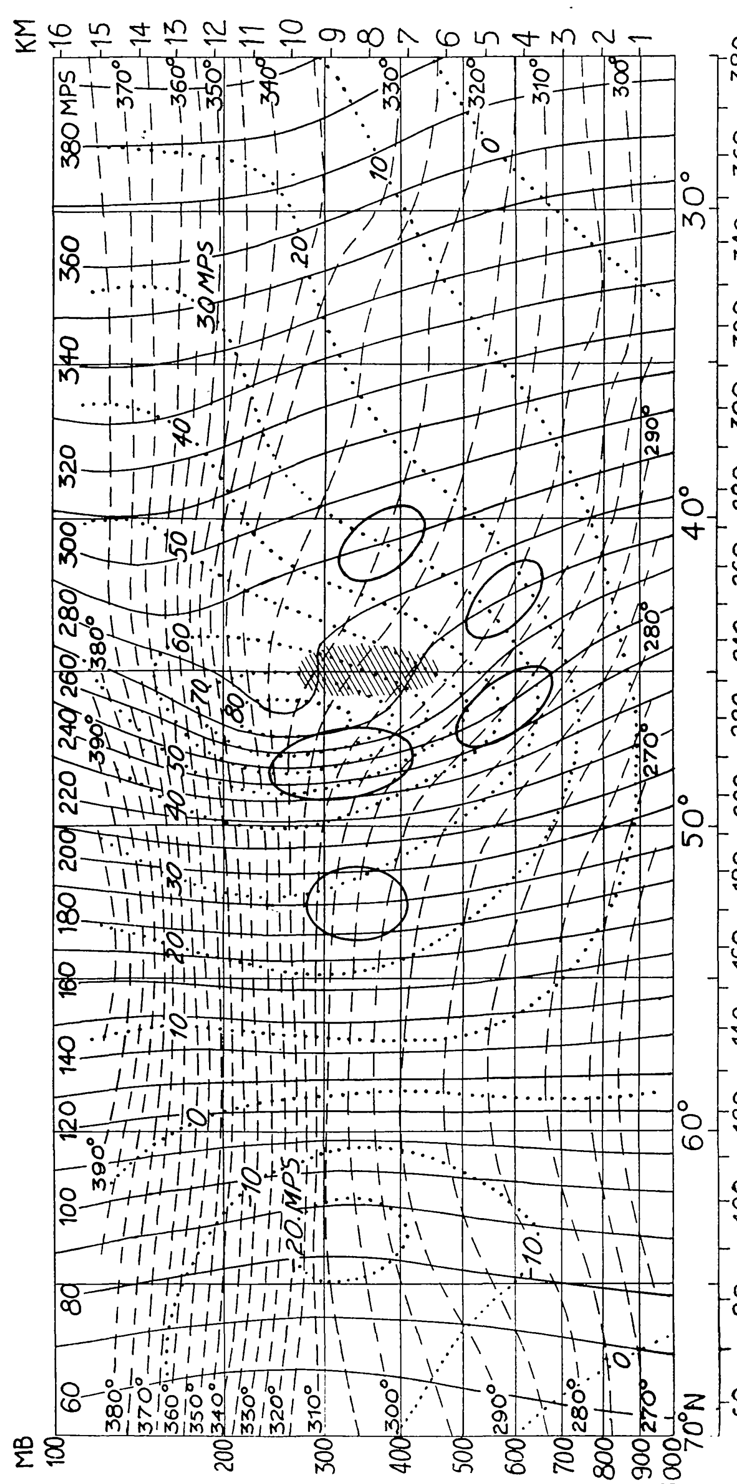

Fig. 16. Meridional cross-section through the atmosphere at 0300 GCT, 30 November 1946, based on observations from North America. The lines of constant zonal geostrophic wind speed (dotted), and the lines of constant potential temperature (dashed) have been copied from Palmén and Nagler (1948). The lines of constant angular momentum (solid) have been constructed by means of eq. (79). These lines are drawn for c/a = a multiple of 10 m sec^{-1}. The scale below the cross-section indicates the angular momentum of the earth's surface, in the same units.

steeper than the slope of the isentropic lines, corresponding to stability with respect to axially symmetric oscillations. This is in agreement with the findings of Palmén and Nagler (1948). Taking into account the difficulties in connection with the application of the theory of the circular vortex to the atmosphere, and possible errors of observation, one should not pay too much attention to the fact that the limit of stability seems to be exceeded within the hatched region. All that can be safely said about this region is that the state is close to the transitional state between stability and instability.

It has been shown in sections 7 and 8 that the meridional streamlines in the vicinity of a point-source of heat or angular momentum are ellipses, the shape and orientation of which depend only on the local structure of the vortex. The ellipses in Fig. 16 are such streamlines constructed by means of eq. (40) in five different points of the cross-section. To obtain the true shape of these ellipses in the meridional plane, the vertical scale of the diagram must be reduced about 160 times. The vertical diameter of the ellipses is therefore in reality very small compared to their horizontal extent, and the major axis will in all cases be orientated approximately along the isentropic lines. The major axis is thus approximately horizontal in nearly barotropic regions, and is sloping along the isentropic lines in baroclinic regions. The ratio of the minor axis to the major axis, which is everywhere a small quantity, will decrease with increasing baroclinity, and tend toward zero as the vortex approaches the transitional state between stability and instability (i. e. as $\delta \rightarrow 0$). Thus there is a tendency for the streamlines to extend along the isentropic lines, and this tendency is particularly pronounced in regions of strong baroclinity. Obviously this tendency will as a rule be preserved also if we add up the fields produced by several point sources of heat and angular momentum. Hence we conclude that even in the case of an arbitrary distribution of sources of heat and angular momentum, there will be a tendency for the meridional streamlines to extend mainly in the direction of the isentropic lines; and this tendency will be particularly pronounced in regions of strong baroclinity.

Since the sources of angular momentum in the free atmosphere presumably are comparatively weak, one should expect that meridional circulations would cause a weakening of the gradient of angular momentum in the meridional plane. In Fig. 16, a region of remarkably weak gradient of angular momentum is found in middle latitudes, south of the west wind maximum. It is not unreasonable to assume that this weak gradient of angular momentum is caused by meridional currents, which in this strongly baroclinic region must have a pronounced tendency to follow the sloping isentropic lines, thus carrying angular momentum from low levels in low latitudes into the upper troposphere in middle latitudes. It seems therefore that the occurance of very high westerly wind velocities in the upper troposphere of the middle latitudes would follow as a necessary result of such meridional circulations.

The weakening of the gradient of angular momentum in a region where meridional circulations are going on, must be accompanied by a strengthening of the gradient of angular momentum in the adjacent regions. Such a strengthening shows up very distinctly in the cross-section north of the west wind maximum. Now, since the inertial stability is proportional to the gradient of the angular momentum, meridional circulations will tend to reduce the inertial stability in the region where these circulations occur, and to increase the inertial stability in adjacent regions. It was pointed out in sections 6, 7, and 8 that the speed of the meridional circulations will increase with decreasing stability. Therefore, a meridional circulation in a certain region of the meridional plane will support itself, by reducing the stability within this region, and suppress meridional circulations in the adjacent regions by increasing the stability in these regions.

This mechanism will be recognized as being analogous to the effect of turbulence on lapse rate. Turbulence in the surface layer will support itself by increasing the lapse rate within this layer, but will cause a reduction of the lapse rate and thus suppress turbulence in the adjacent layer above. This leads to the development of the well known turbulence inversion as a sharp upper boundary of the mixed layer. It is not unreasonable to assume that the analogous mechanism for meridional circulations may lead to the development of an abrupt northern boundary of the meridional circulations, thus explaining the remarkable sharpness of the west wind maximum.[1] North of the west

[1] This analogy was introduced by the Chicago school, however, in the opposite sense, by comparing the mixed surface layer with the polar cap north of the west wind maximum. (Staff members, Department of Meteorology, University of Chicago, 1947).

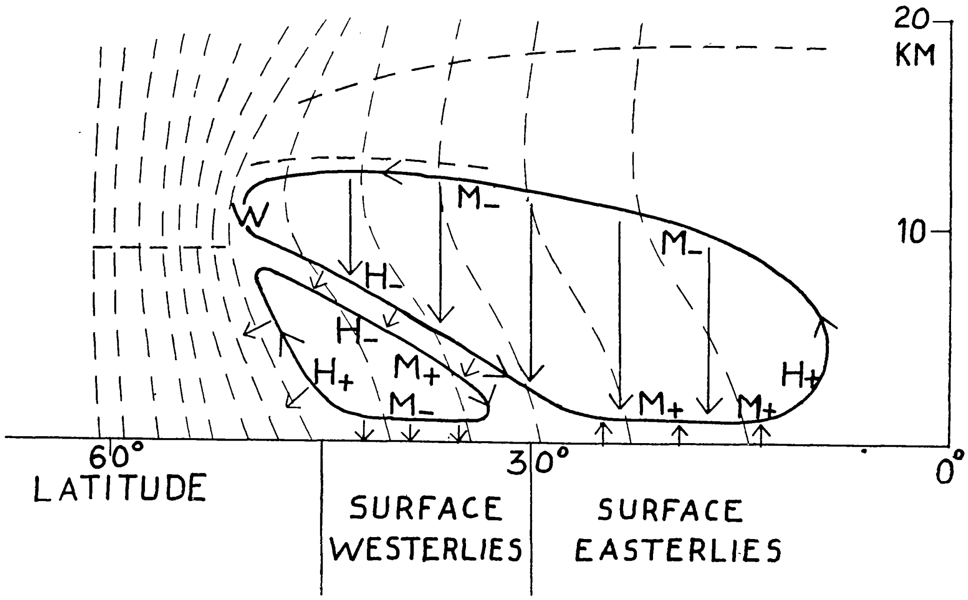

Fig. 17. Schematic meridional cross-section through the atmosphere, showing hypotetical meridional circulations. Solid lines are meridional streamlines. Thin dashed lines are lines of constant angular momentum. Heavy dashed line means tropopause. Arrows indicate turbulent transfer of angular momentum. $M+$ ($M-$) indicate sources (sinks) of angular momentum, and $H+$ ($H-$) sources (sinks) of heat.

wind maximum, meridional circulations are absent owing to the strong (inertial) stability, and thus there is nothing that will destroy the strong gradient of angular momentum, i. e. the strong cyclonic shear of the wind. South of the west wind maximum, the weak stability will favor meridional circulations, and these circulations will in turn maintain a weak gradient of the angular momentum, i. e. a strong anticyclonic shear of the wind.

A proposed scheme of the meridional circulations south of the west wind maximum is shown in Fig. 17. The dashed lines represent lines of constant angular momentum, and the arrows indicate the frictional (turbulent) flux of angular momentum, assumed to take place from strata of higher angular momentum toward strata of lower angular momentum. The corresponding sources and sinks of angular momentum are denoted by M_+ and M_-, respectively. The air is supposed to be heated in lower levels near the equator; and another heat source, due to release of latent heat, is assumed to be located just north of the surface westerlies in middle latitudes. These heat sources are denoted by H_+. Sinks of heat (H_-), due to excess radiation in warm, subsiding air, are indicated in about 40° N. However, the indications of the locations of these heat sources and sinks are based on very crude estimates, and represent an extreme simplification.

From the preceding theory one should expect these sources and sinks of angular momentum and heat to give rise to a meridional circulation with two main circulation cells, as indicated by the solid lines in the diagram. The larger, thermally driven "trade wind cell" extends from the earth's surface in low latitudes to the upper west wind maximum in middle latitudes. This circulation cell carries angular momentum from the surface trades, which constitutes the main source of angular momentum, into the upper westerlies in middle latitudes, where supply of angular momentum is needed for the maintenance of the wind field against frictional dissipation. The smaller indirect "frontal cell" in the north receives angular momentum and energy from the trade wind cell by means of friction. This cell loses angular momentum to the ground in the zone of surface westerlies, which constitutes the main sink of angular momentum. The meridional streamlines are running mainly along the isentropic lines, in agreement with the requirement of our analysis. According to V. Bjerknes (1937), this model is nothing but the old scheme of Ferrel (1856) and James Thomson (1857) in a slightly new version.

The model of meridional circulations, described above, seems to account for the distribution of angular momentum, and hence for the maintenance of the trades and the middle latitude westerlies. However, it is not intended to claim that such meridional circulations is the only mechanism of importance for the maintenance of these wind fields. It is very probable that part of their kinetic energy originates from such wave disturbances that are able to convert thermal energy into kinetic energy.

Appendix.

Let τ denote a symmetric, two-dimensional tensor, whose components

$$\tau = \begin{cases} A\,B \\ B\,C \end{cases}$$

are continuous functions of two independent variables, and let ∇ denote the two-dimensional del-operator in these coordinates. The boundary value problem for the stream function, eqs. (29) and (36), may then be written

$$L(\psi) = \nabla \cdot (\tau \cdot \nabla \psi) = \varphi \qquad \text{in } \gamma, \tag{80}$$

$$\psi = 0 \qquad \text{on } S, \tag{81}$$

where S is the closed boundary curve, and γ the region of the coordinate plane enclosed by S. Eq. (80) is supposed to be of the elliptic type, such that

$$\delta^2 = A\,C - B^2 > 0\,. \tag{82}$$

In the region γ, we choose a point with the position vector $\boldsymbol{r}_0$, and a closed curve s enclosing the point $\boldsymbol{r}_0$. Green's theorem for the perifractic region γ' between S and s gives

$$\begin{aligned} &\int_{\gamma'} G\,\nabla \cdot (\tau \cdot \nabla \psi)\,d\gamma - \int_{\gamma'} \psi\,\nabla \cdot (\tau \cdot \nabla G)\,d\gamma \\ &= \oint_S G\,\boldsymbol{N} \cdot \tau \cdot \nabla \psi\,dS - \oint_S \psi\,\boldsymbol{N} \cdot \tau \cdot \nabla G\,dS \\ &- \oint_s G\,\boldsymbol{n} \cdot \tau \cdot \nabla \psi\,dS + \oint_s \psi\,\boldsymbol{n} \cdot \tau \cdot \nabla G\,dS. \end{aligned} \tag{83}$$

Here $\boldsymbol{N}$ is a unit vector normal to S, pointing out of γ', and $\boldsymbol{n}$ is a unit vector normal to s, pointing toward γ'. We define the Green function G by the following conditions:

$$L(G) = \nabla \cdot (\tau \cdot \nabla G) = 0 \qquad \text{in } \gamma, \text{ except in } \boldsymbol{r}_0, \tag{84}$$

$$G = 0 \qquad \text{on } S, \tag{85}$$

$$\oint_s \boldsymbol{n} \cdot \tau \cdot \nabla G\,ds = 1\,. \tag{86}$$

On account of (84), eq. (86) will hold for any curve s if it holds for one particular curve s. Thus the limit of the integral of (86), when

s contracts toward $\boldsymbol{r}_0$, must also be equal to unity. When τ is continuous, and τ_0 its value in $\boldsymbol{r}_0$, this limit may be written as

$$\lim_{s \to 0} \oint_s \boldsymbol{n} \cdot \tau_0 \cdot \nabla G \, ds = 1. \tag{87}$$

This equation, which may replace (86), shows the nature of the singularity of the Green function in the point $\boldsymbol{r}_0$.

In virtue of (84), (85) and (81), eq. (83) becomes

$$\int_{\gamma'} G \varphi \, d\gamma = -\oint_s G \boldsymbol{n} \cdot \tau \cdot \nabla \psi \, ds + \oint_s \psi \boldsymbol{n} \;\; \tau \cdot \nabla G \, ds. \tag{88}$$

If we let the curve s contract toward the point $\boldsymbol{r}_0$, the second integral on the right tends toward ψ_0 (the value of ψ in $\boldsymbol{r}_0$) on account of (87), and the first integral on the right tends toward zero (this is seen if we let the curve s be a curve $G =$ constant). Hence we obtain

$$\psi_0 = \int_\gamma G \varphi \, d\gamma, \tag{89}$$

which is identical with eq. (37).

It was stated in section 6 that the function G_1, defined by (38), constitutes the principal part of G near the singular point. We shall show that G_1 actually fulfills eq. (87). Apart from an additive constant, G_1 may be written

$$G_1 = \frac{1}{2\pi\delta_0} \ln \left[(\boldsymbol{r} - \boldsymbol{r}_0) \cdot \tau_0^{-1} \cdot (\boldsymbol{r} - \boldsymbol{r}_0) \right]^{\frac{1}{2}}, \tag{90}$$

where subscripts "0" denote values in the point $\boldsymbol{r}_0$, and τ^{-1} is the inverse tensor

$$\tau^{-1} = \begin{cases} C/\delta^2, & -B/\delta^2 \\ -B/\delta^2, & A/\delta^2 \end{cases} \tag{91}$$

From (90) we obtain

$$\nabla G_1 = \frac{1}{2\pi\delta_0} \frac{\tau_0^{-1} \cdot (\boldsymbol{r} - \boldsymbol{r}_0)}{(\boldsymbol{r} - \boldsymbol{r}_0) \cdot \tau_0^{-1} \cdot (\boldsymbol{r} - \boldsymbol{r}_0)}, \tag{92}$$

and consequently, for any curve s,

$$\oint_S \boldsymbol{n} \cdot \tau_0 \cdot \nabla G_1 \, d s = \frac{1}{2 \pi \delta_0} \oint_S \frac{(\boldsymbol{r} - \boldsymbol{r}_0) \cdot \boldsymbol{n} \, d s}{(\boldsymbol{r} - \boldsymbol{r}_0) \cdot \tau_0^{-1} \cdot (\boldsymbol{r} - \boldsymbol{r}_0)}$$

$$(93) \qquad = \frac{1}{2 \pi \delta_0} \int_0^{2\pi} \frac{(\boldsymbol{r} - \boldsymbol{r}_0)^2 \, d \theta}{(\boldsymbol{r} - \boldsymbol{r}_0) \cdot \tau_0^{-1} \cdot (\boldsymbol{r} - \boldsymbol{r}_0)}$$

$$= \frac{\delta_0}{2 \pi} \int_0^{2\pi} \frac{d \theta}{C_0 \cos^2 \theta - 2 B_0 \cos \theta \sin \theta + A_0 \sin^2 \theta} = 1 ,$$

where θ is the argument angle in a polar coordinate system centered at $\boldsymbol{r}_0$. Thus G_1 fulfills eq. (87), and the assertion is proved.

List of symbols.

R	distance from the axis,
z	height above a plane normal to the axis,
t	time,
Φ	gravity potential,
p	pressure,
α	specific volume,
T	absolute temperature,
σ	entropy,
μ	defined on p. 22,
c_p	specific heat at constant pressure,
ω	absolute angular velocity,
c	angular momentum per unit mass,
$\dot{R}, \dot{p}$	individual rate of change of R and p,
D/dt	individual derivative,
ψ	stream function of the meridional motion, defined on p. 26,
Q	heat received by the fluid per unit mass and unit time,
χ	frictional torque,
g_z	component parallel to the axis of the acceleration of gravity,
$A, B, C, E, F,$	coefficient functions, defined on p. 27,
S	boundary curve in the merdional plane,
γ	region enclosed by S,
G	Green function,
ν_g	frequency of gravitational oscillations,
ν_i	frequency of inertial oscillations,
H	enthalpy,
h	enthalpy per unit mass,
K	kinetic energy of the vortex motion,
Ψ	defined on p. 46,
$\nabla R, \nabla p$	fictitious meridional displacements, p. 47,
a	radius of the earth,
φ	latitude,
u	relative zonal wind speed.

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