ARKIV FÖR MATEMATIK, ASTRONOMI OCH FYSIK.
BAND 2. N:o 11.

On the Influence of the Earth's Rotation on Ocean-Currents.

By

V. WALFRID EKMAN.

With 1 plate and 10 figures in the text.

Communicated May 10th 1905 by O. PETTERSSON and V. BJERKNES.

The influence of the earth's rotation on the currents in the Atmosphere (and the Ocean) was pointed out long ago by Hadley, Coriolis and Ferrel. It has however until lately had very little influence upon theories of ocean-currents except as far as concerns such *free* currents, as continue to move by their own inertia after the moving force has ceased; and the reason obviously is that the »deflecting forces» due to the small velocities of the currents were supposed to be too small compared to the moving forces themselves[1]. Thus ZÖPPRITZ' theory of drift-currents[2] which leaves the earth's rotation out of account, was generally accepted for a long time, and MOHN in his system[3] though attaching much weight to the effect of the deflecting forces, based his calculation essentially on the results of Zöppritz' theory.

V. BJERKNES was the first to give[4] to the laws of motion of the atmosphere and hydrosphere, a form clearly in-

[1] See for instance: BOGUSLAVSKI und KRÜMMEL Ozeanographie II Seite 363. Prof KRÜMMEL points out in the same work (p. 393), that the drift-currents really deviate to the right of the wind-direction in the North Atlantic and to the left in the South Atlantic; but he is inclined to explain this not as an effect of the earth's rotation but essentially as a result of other cooperating circumstances.

[2] K. ZÖPPRITZ: Hydrodynamische Probleme in Beziehung zur Theorie der Meeresströmungen. Wied. Ann. III (1878), Seite 582.

[3] H. MOHN: »Nordhavets Dybder, Temperaturer og Strömninger». Den Norske Nordhavs-expedition 1876—1878. II. Christiania 1887.

[4] V. BJERKNES: Cirkulation relativ zu der Erde. Öfversikt af Kongl. Vet. Akad. Förhandl. 1901. N:o 10. Stockholm.

dicating the importance of the earth's rotation also upon the *forced* currents, and in particular upon those caused by differences of density. The only formal difficulty to overcome if his theorem should have its full applicability, is the quantitative estimation of the influence of friction.

When the currents are driven by the wind, the problem is in a way a simpler one; for in this case the friction is at the same time the only moving force, and the coefficient of friction is therefore eliminated from several questions. On studying the observations of wind and ice-drift taken during the drift of the »Fram», FRIDTJOF NANSEN found that the drift produced by a given wind did not, according to the general opinion, follow the wind's direction but deviated 20°—40° to the right[1]. He explained this deviation as an obvious consequence of the earth's rotation; and he concluded further that the water-layer immediately below the surface must have a somewhat greater deviation than the latter and so on, since every water-layer is put in motion by the layer immediately above, sweeping over it like a wind. It might therefore be assumed *a priori*, that the current would at some depth run even in the *opposite* direction to the surface-current; and there would consequently be a limit to the capability of the wind in generating currents.

On Professor Nansen's suggestion, I investigated the problem mathematically[2] and found results confirmatory of his opinion above mentioned. Further, it was proved that independently of any other circumstances but the geographical latitude, a wind-current would become practically fully developed a very short time after the rise of the generating wind — in a day or two outside the tropics. In this investigation the very important influence of continents, differences of density of the water, and other complicating circumstances, were expressly left out of account.

In the present communication some of these restrictions will be removed, and particularly the influence of the conti-

[1] FRIDTJOF NANSEN: Oceanography of the North Polar Basin. The Norwegian North Polar Expedition 1893—96. Scientific Results Vol. III, N:o 9. Kristiania 1902.

[2] V. WALFRID EKMAN: »Om jordrotationens inverkan på vindströmmar i hafvet.» Nyt Magazin for Naturvidenskab. B. 40. H. 1. Kristiania 1902. As it would for some reasons be inconvenient here to refer to this paper, the subject will be treated independently of it, right from the beginning.

nents and of neighbouring currents etc. will be examined. By these additions the result of the theory is brought markedly towards agreement with MOHN's system, though the very essential differences left, are still quite obvious.

The calculation showed, as might be expected, a decided tendency on the part of the surface-current to follow the direction of the shore-lines though with a deviation more or less to a direction 45° to the *right* of the wind (in the northern hemisphere). Besides this modifying influence on the surface-currents, the continents have another more important effect. The surface-current, which itself fluctuates even with the more transient changes of the wind, will as a rule by the accumulation of water (towards a coast for instance) give rise to a more steady deep current running in this particular case, parallel to the coast and with a velocity uniform almost right down to the bottom. The velocity of this »*mid-water-current*» is proportional simply to the wind-component in the direction of the coast and may, independently of the depth, be even more than half the velocity of the surface-current. The time required for its development has to be counted in days or months according to the depth of the sea and the breadth of the current; and the midwater-current though in a way depending upon the *average* wind, will in any case to some extent follow its seasonal variations. Neither geological periods — which according to Zöppritz' theory would be required for the establishment of the currents as they are at present — nor even centuries or decades, would come into question at all. Below the »midwater-current», and with a velocity-component perpendicularly to this, is a *bottom-current* of the same depth as the surface-current. This bottom-current compensates the flow towards or from land in the surface-current.

One of the greatest difficulties in following up the theory of ocean-currents quantitatively, is that we do not know the magnitude of the mutual reaction between the waterlayers; of which in the case of quite regular motion, the coefficient of viscosity would be a measure. As has been pointed out by various authors, particularly by BOUSSINESQ, HELMHOLTZ and O. REYNOLDS. this reaction is enormously greater than it would be in the case of regular motion owing to the irregular formation of vortices; and if we wish to calculate the large regular currents without the irregular and incalculable vortex-motion, it is therefore necessary to introduce a virtual value of the coefficient of friction μ. BOUSSINESQ has shown that for wide tubes, canals etc. it is possible in this way to get fairly exact results, by giving to μ a value varying in a suitable way from one water-layer to another. For our purpose it will as a rule be sufficient to assume a constant value for μ, though different values may have to be given to this constant under different circumstances. It has proved very convenient to substitute for the coefficient of friction μ another quantity D. This quantity which is proportional to $\sqrt{\mu}$ and inversily proportional to the square root of the sine of the latitude, has a linear dimension; it may be called the »Depth of Wind-current» or more generally »*Depth of frictional influence*». It enters in a simple and easily understood manner into the equations and seems to be in general, very useful in the theory of ocean-currents. An attempt is made in section IV of this paper to calculate a preliminary value of this quantity. By help of the quantity D it is also possible in a very simple way, duly to consider the effect of friction on convection-currents when calculated by means of V. BJERKNES' above mentioned theorem.

Owing to the pressure of other work it has not been possible at present, to follow up the mathematical results to a definite theory of the actual ocean-currents, and only a few remarks are made in that direction. Various modifications of the mathematical problems are also excluded here, since they are most conveniently examined simultaneously with the practical applications.

On the other hand the author wished to give a short account of an experiment which was made by the late Prof. C. A. BJERKNES at Christiania. This wish was responded to by his son Prof. V. BJERKNES, and through the kindness of Prof. SCHIÖTZ the author therefore got admission to C. A. Bjerknes' laboratory, and had the opportunity of repeating his experiments.

Before entering upon the subject the author desires to record his sincere thanks to his friend Dr. Charles J. J. Fox, who most kindly revised the English.

I. Currents caused by the wind and the earth's rotation, alone.

To get a clear notion of the influence of the earth's rotation — and also of the friction in the water — consider first the simplest possible case of a wind-current. Imagine a large ocean of uniform depth and without differences of density affecting the motion of the water. The influence of neighbouring ocean-currents and continents may be left out of account; and it is therefore assumed that water can freely enter into or flow from the region considered. Finally the curvature of the globe may be disregarded within this region, and the sea-surface treated as plane. Suppose the sea-surface to be impelled by a steady and uniform wind equal in strength and in direction over the whole region; and that these circumstances have lasted so long that a practically stationary state of motion has become established. It then follows by symmetry that the motion will consist of a gliding of the water-layers one over the other much as a bundle of thin boards might be imagined; the direction and velocity of the motion being uniform within each horizontal layer.

The equations of motion of the water, the latter being regarded as incompressible, are

$$(1)\quad \begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z} = X - \dfrac{1}{q}\dfrac{\partial p}{\partial x} + \dfrac{\mu}{q}\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right) \\ \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z} = Y - \dfrac{1}{q}\dfrac{\partial p}{\partial y} + \dfrac{\mu}{q}\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial^2 v}{\partial z^2}\right) \\ \dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z} = Z - \dfrac{1}{q}\dfrac{\partial p}{\partial z} + \dfrac{\mu}{q}\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2}\right) \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0, \end{cases}$$

where

u, v, w = the velocity components in the directions of x, y, z
X, Y, Z = the components of extraneous forces per unit mass
q = the density of the fluid
p = the pressure
μ = the coefficient of viscosity, and
t = the time.

Assume the axis of z to be taken vertical (positive downwards) and the axis of x and y horizontal, the positive direction of y being $90°$ to the *left* of the positive direction of x. Then it follows by symmetry and from the last one of the equations (1) that

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \text{ and } w$$

are all identically equal to zero.

Since no real extraneous force but gravity, is to be taken into account, X and Y are only the horizontal components of the deflective force due to the earth's rotation:

$$X = 2\, v\omega \sin \varphi;\ Y = -2\, u\omega \sin \varphi,$$

where ω is the angular velocity of the earth 0,0000729, and φ is the latitude.

The two last equations (1) are consequently unnecessary, and the two first become

$$(2)\qquad \begin{aligned} \frac{\partial u}{\partial t} &= \quad 2\, v\omega \sin \varphi + \frac{\mu}{q}\frac{\partial^2 u}{\partial z^2} \\ \frac{\partial v}{\partial t} &= -2 u\omega \sin \varphi + \frac{\mu}{q}\frac{\partial^2 v}{\partial z^2}. \end{aligned}$$

In the present case of stationary motion, the terms on the left hand side vanish, and z is now the only independent variable. With the notation

$$a = +\sqrt{\frac{q\omega \sin \varphi}{\mu}},$$

equations (2) then take the form

$$(3)\qquad \frac{d^2u}{dz^2} + 2\, a^2 v = 0;\ \frac{d^2 v}{dz^2} - 2\, a^2 u = 0,$$

and the general solution is obviously

$$(4)\qquad \begin{aligned} u &= C_1 e^{az} \cos (az + c_1) + C_2 e^{-az} \cos (az + c_2) \\ v &= C_1 e^{az} \sin (az + c_1) - C_2 e^{-az} \sin (az + c_2), \end{aligned}$$

C_1, C_2, c_1, c_2, being arbitrary constants and $e = 2{,}718\ldots.$ As we want real values of a, we must assume φ to be positive; and the results which follow are consequently applicable only to the *northern* hemisphere. It is however easy to see in what manner a particular result must be altered, to be applicable to the southern hemisphere. As a matter of fact the same equations hold unaltered in this latter case if the axis of positive y be drawn 90° to the *right* of the axis of positive x.

Our solution (4) takes the simplest form if the ocean is assumed to be *infinitely deep, i. e.* on the assumption that the velocity is zero for z infinite. (Practically this only implies that the sea-bottom is below the greatest depth at which the wind is able to produce any sensible motion; it will be shown below, that this depth is very small compared to actual ocean depths and hardly exceeds 200 or 300 meters). C_1 is then zero and equations (4) are reduced to

$$u = C_2 e^{-az} \cos\ (az + c_2)$$
$$v = -C_2 e^{-az} \sin\ (az + c_2).$$

On differentiating we get

$$\frac{du}{dz} = -a\sqrt{2}C_2 e^{-az} \sin\ (az + c_2 + 45^0)$$
$$\frac{dv}{dz} = -a\sqrt{2}C_2 e^{-az} \cos\ (az + c_2 + 45^0).$$

Assuming that the tangential pressure of the wind on the sea-surface, is T and directed along the positive axis of y, C_2 and c_2 are determined by

$$\mu\left(\frac{du}{dz}\right)_{z=0} = 0;\ -\mu\left(\frac{dv}{dz}\right)_{z=0} = T.$$

If further V_0 be the absolute velocity of the water at the surface, $V_0 = C_2$; and we get

$$(5)\qquad \begin{aligned} u &= V_0 e^{-az} \cos\ (45^0 - az) \\ v &= V_0 e^{-az} \sin\ (45^0 - az) \\ V_0 &= \frac{T}{\mu a\sqrt{2}} = \frac{T}{\sqrt{2\mu q\omega \sin\varphi}} \end{aligned}$$

The direction of the tangential pressure T — *i. e.* of the axis of y — is of course the direction of the wind-velocity *relative to the water*, as *e. g.* when determined from a ship drifting with the surface-water[1].

[1] The velocity of the air should then strictly be determined infinitely near to the water-surface. The direction of the wind (relative to

Equations (5) then show that in the northern hemisphere the *drift-current at the very surface will be directed 45° to the right of the velocity of the wind* (relative to the water)[1]. In the southern hemisphere it is directed 45° to the *left*. And this angle further increases uniformly with the depth, four right angles for each time the depth increases by $2\pi/a$. At the same time the velocity of the water decreases with the depth to $e^{-2\pi} = 1/535$th part for each time its direction rotates four right angles. The direction and velocity of the

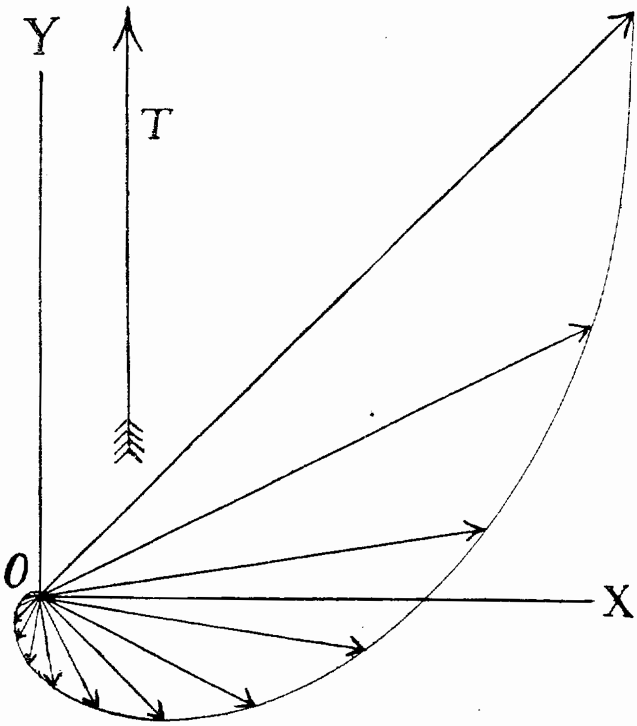


Fig. 1.

current at different depths are represented by the arrows in Fig. 1 above; the longest arrow refers to the surface, the

the water) does not however vary appreciably with the height, within the limits at which it is usually measured and right down to the water-surface; and the coincidence of the wind's direction and the direction of tangential pressure therefore holds true just as well if the former be reckoned at some distance above the sea-level. By the direction and velocity of the wind will therefore always be understood those given at the height usual in meteorological observations, since there is of course no particular reason at all for any other definition.

With this remark a criticism put forward by Dr. FILIP ÅKERBLOM (Recherches oceanografiques, Upsala Universitets årsskrift 1903) has been answered.

[1] The angle between the direction of the surface-current and that of the *absolute* wind-velocity is somewhat smaller; but as the velocity of the water is as a rule much smaller than that of the wind — a few hunderths of the latter only — the difference will in any case be unimportant.

next one to the depth $z = \pi/10\,a$, and the other ones to 2, 3, 4 etc. times this depth. These arrows, if conceived as drawn in position at their respective depths, would then appear in the form of a spiral staircase, the breadth of the steps decreasing rapidly downwards. This stair-case would actually become microscopical before the first winding is completed.

According to what has been said above, it would be to a certain extent uncertain what to denote as the depth of the current produced by the wind; an exact definition of it will however be very useful in the following investigation. By the »*Depth of Wind-current*» is therefore to be understood the depth down to that level where according to (5) the velocity of the water is directed opposite to the velocity at the surface[1]. With this definition it is

$$(6) \qquad D = \frac{\pi}{a} = \pi \sqrt{\frac{\mu}{q\omega \sin \varphi}}.$$

If the velocity components parallel to the surface velocity be considered, Fig. 1 shows that there is an upper current between the surface and the level $z = {}^1/_2 D$, and below this down to the level $z = D$ a comparatively weak current running in the opposite direction. Perpendicularly to the surface-velocity the current has the same direction from the surface and right down to the level $z = D$. The total flow of water in the directions of x and y is respectively

$$(7) \qquad S_x = \int_0^\infty u dz = \frac{V_0 D}{\pi \sqrt{2}} = \frac{T}{2\, q\omega \sin \varphi}; \quad S_y = \int_0^\infty v dz = 0,$$

i. e. the total momentum of the current generated by the wind, is directed one right angle to the right of the wind itself. The physical explanation of this fact is very simple. At infinite depth (or practically, at depths exceeding $z = D$) the velocity and consequently the friction between the waterlayers, is zero; the whole mass of water measured from the surface down to a great depth, is then impelled by no other external forces than the wind and the deflecting force due to the earth's

[1] The author has before (l. c. p. 2) proposed to denote *half* this depth as the depth of the surface current generated by the wind. On following up the theory with attention to the influence of continents etc., it has however appeared that the choice now made is more suitable.

rotation. When the motion is stationary, these forces must be equal and opposite in direction; and since the deflecting force is directed one right angle to the right of the Centre of gravity's direction of motion, the latter must be directed one right angle to the right of the wind. It is then, obvious that the direction and magnitude of the flow depend only upon the tangential pressure T and are quite independent of the value of μ or of whether μ is a constant or alters with the depth.

The above-mentioned result, according to which the surface-current's deflection from the wind-direction, is invariably 45°, seems rather strange; one would indeed expect the earth's rotation to have less influence on the currents, the smaller its vertical component $\omega \sin \varphi$. The deflecting force does not however depend on $\omega \sin \varphi$ alone, but equals twice the product of this multiplied by the mass and the velocity of the current; and as the two latter factors increase towards the equator as $(\sin \varphi)^{-1/2}$, the product remains constant. On the other hand the *abrupt* change of the angle of deflection, from 45° to the right to 45° to the left, which theoretically should take place on passing from the northern to the southern hemisphere, has of course no correspondence with actual reality. To restore continuity it would be sufficient to remark that the depth D down to which the effect of the wind is noticeable, increases without limit on moving towards the equator and finally will exceed the depth of the ocean; so that the latter cannot at all be regarded as infinitely deep. It will be seen from the subsequent examination of the case of finite depth, that on this account the angle of deflection of stationary wind-currents would begin to decrease in the neighbourhood of the equator, and be zero on the equator itself. The real reason why the solution contained in (5) cannot be applied near to the equator, is however that wind-currents there cannot be treated as *stationary*. For as the depth and velocity of a stationary wind-current would be very great there, it would require a long time for it to attain its final velocity, and the water might have by then moved into regions with other winds or other different conditions before the stationary motion has had time to become established.

If the Depth of Wind-current be 100 say, at the poles, it would be about 108 at 60° latitude, 141 at 30°, 240 at

10°, and infinite at the equator, where however our solution as mentioned, does not hold true. The velocity of the surface-current varies in the same ratio. On the other hand the Depth of Wind-current would be independent of the strength of the wind. This result, which seems very surprising, must actually be to some extent modified; for as μ probably increases with the violence of the motion, this circumstance will cause an increase of the current's depth with the strength of the wind.

As was mentioned at the beginning, the determination of the friction in the water is a very serious difficulty in the numerical computation of the theory of ocean-currents; and this is particularly the case in calculating the depth of wind-currents. When the value $\mu = 0{,}014$ *C. g. s.* units, found experimentally for the case of quite regular motion, is introduced into (6) it gives the absurd result

$$D = \frac{44}{\sqrt{\sin\,\varphi}}.$$

I. e. the influence of the wind in producing ocean-currents should be restricted to a surface layer 44 cm. thick in the polar regions and 70 cm. under the tropics; and at half these depths the water should run perpendicularly to the direction of motion in the surface-water. It is however instructive to compare this result with ZÖPPRITZ' theory, since the latter is based upon exactly the same assumptions with regard to the magnitude of the friction etc., only that no account is taken of the influence of the earth's rotation. He found that the influence of the average winds would in the course of geological time-periods extend right down to the bottom, and the wind-current would finally run with a velocity proportional to the height above sea-bottom and in the same direction from bottom to surface.

The mutual action of the water-layers upon one-another, is however owing to the irregular formation of eddies, incomparably more intense than it would be if due to friction alone when the motion itself is quite regular. It is therefore necessary to introduce a virtual value of μ, much greater than its real value $\mu = 0{,}014$; and according to (6) the depth of the wind-current will be great in proportion to the square root of this virtual value.

It will often be convenient in what follows to substitute for μ the quantity D, which with a name corresponding to its more general application, may be called the »*Depth of frictional influence*». The formulae will then often appear in a simplified form, and furthermore this quantity is much more intimately related, to the physical reality which we are studying than is the virtual value of μ. It is then

$$\mu = \frac{D^2 q\omega \sin \varphi}{\pi^2}.$$

The virtual value of μ — and the quantity D — is probably very different under different circumstances; and the only means of getting information on this important subject would be by current-measurements and other observations, systematically carried out under different circumstances and in different parts of the sea, in bays and lakes etc. It will be shown in section IV in what way this could be done; and a rough calculation will also be made of the order of magnitude of the quantity D. In the meantime it may be mentioned that D will vary about as the wind velocity, and that according to calculation 75 m. would probably be a very common mean value of D.

In the case of an ocean of *finite uniform depth d*, the constants C_1, C_2, c_1, c_2 in (4) p. 6 will satisfy the equations

$$u = v = 0 \text{ for } z = d.$$

It is convenient in this case to reckon depths from the sea-bottom instead of from the surface. If then

$$\zeta = d - z$$

denotes the distance from the bottom, equations (4) become

$$u = \tfrac{1}{2} C[e^{a\zeta} \cos (a\zeta + c) - e^{-a\zeta} \cos (a\zeta - c)]$$
$$v = \tfrac{1}{2} C[e^{a\zeta} \sin (a\zeta + c) - e^{-a\zeta} \sin (a\zeta - c)].$$

If as before we assume that the wind impels the water with a tangential pressure T in the direction of y, and if we adopt the usual and very convenient notation

$$\text{Cosh } x = \frac{e^x + e^{-x}}{2}; \text{ Sinh } x = \frac{e^x - e^{-x}}{2},$$

the final expressions for u and v become[1]

$$(8)\quad \begin{cases} u = A \operatorname{Sinh} a\zeta \cos a\zeta - B \operatorname{Cosh} a\zeta \sin a\zeta \\ v = A \operatorname{Cosh} a\zeta \sin a\zeta + B \operatorname{Sinh} a\zeta \cos a\zeta \\ A = \dfrac{TD}{\mu\pi} \dfrac{\operatorname{Cosh} ad \cos ad + \operatorname{Sinh} ad \sin ad}{\operatorname{Cosh} 2ad + \cos 2ad} \\ B = \dfrac{TD}{\mu\pi} \dfrac{\operatorname{Cosh} ad \cos ad - \operatorname{Sinh} ad \sin ad}{\operatorname{Cosh} 2ad + \cos 2ad}. \end{cases}$$

The total flow in the directions of x and y are respectively

$$(9)\quad \begin{aligned} S_x &= \int_0^d u d\zeta = \frac{TD^2}{2\mu\pi^2} \cdot \frac{\operatorname{Cosh} 2ad + \cos 2ad - 2 \operatorname{Cosh} ad \cos ad}{\operatorname{Cosh} 2ad + \cos 2ad} \\ S_y &= \int_0^d v d\zeta = \frac{TD^2}{\mu\pi^2} \; \frac{\operatorname{Sinh} ad \sin ad}{\operatorname{Cosh} 2ad + \cos 2ad}. \end{aligned}$$

The latter equation shows the peculiar fact, that a wind would in certain cases provoke a flow of water, directed at a small angle *against the wind* (the angle is greatest in the case $d = {}^5/_4 D$ and is then 1,5 degrees only). If for instance the wind blows from land, the effect will then be a *raising* of the sea-level towards land; the earth's rotation acts as a veritable pulley, reversing the direction of the force. The velocity in the surface must of course always have a component *with* the wind; otherwise the latter would check the motion instead of sustaining it.

The angle α between the wind and the surface-current, is not exactly 45°, when the depth is finite. On the contrary

$$\tan \alpha = \left(\frac{u}{v}\right)_{\zeta = d} = \frac{\operatorname{Sinh} 2ad - \sin 2ad}{\operatorname{Sinh} 2ad + \sin 2ad}; \left(2ad = \frac{2\pi d}{D}\right),$$

and the angle of deflection α consequently depends on the ratio between the depth of the sea d and the Depth of Wind-

[1] The formulae:
$2(\operatorname{Cosh}^2 x - \cos^2 x) = \operatorname{Cosh} 2x - \cos 2x$
$2(\operatorname{Cosh}^2 x - \sin^2 x) = \operatorname{Cosh} 2x + \cos 2x = 2(\operatorname{Cosh}^2 x \cos^2 x + \operatorname{Sinh}^2 x \sin^2 x)$
$e^x \cos 2x + e^{-x} = 2(\operatorname{Cosh} x \cos^2 x - \operatorname{Sinh} x \sin^2 x)$,
which are sometimes made use of in the following calculations, are easily verified. The same may be said of the formulae for addition of arguments, for derivation etc. which are analogous to the corresponding formulae in the case of trigonometric functions.

current D. If d/D is a small fraction, α is small and the current goes nearly in the direction of the wind. As the depth increases, α is alternately smaller and greater than 45°. Thus for instance $\alpha = 21°,5$ for $d = 0,25\ D$, $\alpha = 45°$ for $d = 0,5\ D$, $\alpha = 45°,5$ for $d = 0,75\ D$, and $\alpha = 45°$ for $d = D$. When d is

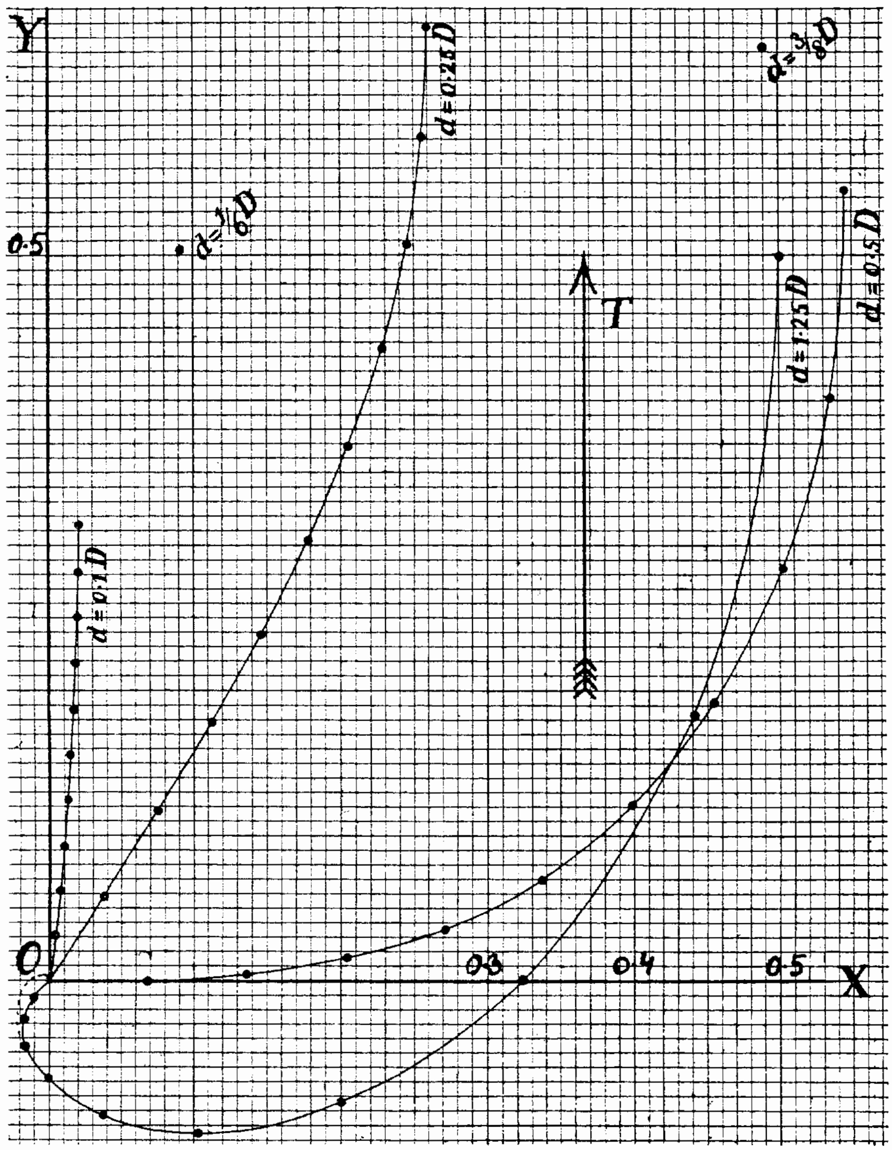


Fig. 2.

greater than D, the deviations from the mean value $\alpha = 45°$ are quite insignificant, and the motion takes place almost exactly as on the deep sea.

The curves in Fig. 2 illustrate the character of wind-currents in seas of different depths $d/D = 0,1,\ 0,25,\ 0,5,\ 1,25$.

The arrows are not drawn, merely the end-points are denoted and then connected by a curve. The points to the number of 10 on each curve, thus represent the velocities at the levels $0{,}1\ d$, $0{,}2\ d$ ..... above the sea-bottom. A dotted line represents that part of the curve for $d = 2{,}5\ D$ which does not fall in with the curve for $d = 1{,}25\ D$; the points are of course not common to these two curves even where the curves themselves coincide. The surface velocities for $d = {}^1/_6\,D$ and $d = {}^3/_8\,D$ are represented by solitary points. The strength of the wind is the same in all cases, namely $T = \mu\pi/D$ and is directed along the axis of y.

The results concerning stationary wind-currents given above would obviously be of little value, as long as we do not know whether or not such currents upon the whole exist, and under which conditions they may exist. It might indeed be possible that a very long time would be necessary for the wind to give to the large quantities of water such great velocities as it is able to maintain against the small friction; and that a wind-current would consequently be only on the rise, when the wind generating it had already changed its direction or the water had reached regions with other wind-conditions. Thus for instance the enormous times Zöppritz has found to be necessary for the penetration of the current even to very moderate depths, are famous in oceanographical literature; though certainly his values are meaningless because they are based on the value $\mu = 0{,}014$. Actually however the value of μ does not at all influence the answer of the question which particularly interests us; and this depends upon the circumstance that the depth to which the wind-current is owing to the earth's rotation limited, increases with μ at the same rate as the velocity with which the current spreads downwards in its first rising — in any case, if the water may be regarded as infinite, or only large compared to D.

Suppose that the water is initially motionless, and that at the time $t = 0$, a steady wind (tangential pressure $= T$) suddenly begins to blow in an invariable direction. It then follows from equations (2) simply by means of considerations of dimensions, that the velocity components u and v at any time t, may be given in the form

$$u=\frac{T}{\sqrt{\mu q}}\Phi_1\left(\overline{\omega}, z\sqrt{\frac{q}{\mu}}, t\right)$$

$$v=\frac{T}{\sqrt{\mu q}}\Phi_2\left(\overline{\omega}, z\sqrt{\frac{q}{\mu}}, t\right),$$

where $\overline{\omega}=\omega \sin \varphi$.

These equations show that by increasing μ in the ratio m a new motion similar to the actual motion is obtained, in which the vertical distances z are increased in the ratio $\sqrt{m}$ the velocities diminished in the ratio $1:\sqrt{m}$ while the time intervals remain *unaltered. I. e.: the time required for a wind-current to attain* $^1/_2$, $^9/_{10}$ *etc. of its final depth or its final velocity, is independent of the value of the coefficient of friction* μ. By some rough aproximations the writer has found that the current generated by a steady wind will in high latitudes become practically fully developed in direction, velocity and depth in about 12 or 24 hours; on advancing towards the equator this time would increase inversely as the sine of the latitude[1].

Dr. I. FREDHOLM subsequently found the exact mathematical solution (equations 10 below) of the problem, and he has been kind enough to allow me to publish it here. Suppose as before that the water is initially motionless, and that at the time $t=0$ a steady wind (tangential pressure $=T$) begins to blow in the direction of the positive y-axis. The depth of the water is assumed to be infinite. Then the velocity components u and v at any depth z and at any time t are given by[2]

(10)
$$u=\frac{T}{\sqrt{q\mu\pi}}\int_0^t \sin\, 2\,\overline{\omega}\zeta\,\frac{e^{-\frac{qz^2}{4\,\mu\zeta}}}{\sqrt{\zeta}}\,d\zeta$$

$$v=\frac{T}{\sqrt{q\mu\pi}}\int_0^t \cos\, 2\,\overline{\omega}\zeta\,\frac{e^{-\frac{qz^2}{4\,\mu\zeta}}}{\sqrt{\zeta}}\,d\zeta,$$

where $\overline{\omega}=\omega \sin \varphi$.

---

[1] EKMAN l. c. p. 2.

[2] It is immediately seen that (10) satisfies equations (2) as well as two of the boundary conditions namely $u=v=0$ for $t=0$ and $\frac{\partial u}{\partial z}=0$ for

With the substitution

$$\tau = t\frac{\overline{\omega}}{\pi}$$

and on account of (6), this may be written in the form

$$(11)\qquad \begin{aligned} u &= \frac{\pi T}{qD}\int_0^\tau \frac{\sin 2\pi\zeta}{\sqrt{\zeta}} e^{-\frac{\pi z^2}{4D^2\zeta}} d\zeta \\ v &= \frac{\pi T}{qD\overline{\omega}}\int_0^\tau \frac{\cos 2\pi\zeta}{\sqrt{\zeta}} e^{-\frac{\pi z^2}{4D^2\zeta}} d\zeta. \end{aligned}$$

Here the time τ is expressed in a unit which at the poles is 12 siderial hours and in general is 12 siderial hours divided by sin φ. It is the time in which the plane of oscillation of a pendulum turns round 180 degrees; it may be conveniently and shortly designated 12 *pendulum-hours*[1]. A pendulum-hour is then, outside the tropics, of the same order of magnitude as an hour, and stands to it nearly in the ratio 1 : sin φ (φ taken positively in the southern as well as in the northern hemisphere).

The gradual development of the steady wind-current may be clearly understood from the *hodographs* of the motion at

---

$z=0$. Only $\frac{\partial v}{\partial z}$ for $z=0$ remains to be calculated. The second equation (10) gives

$$\frac{\partial v}{\partial z} = -\frac{T}{2\mu}\sqrt{\frac{q}{\mu\pi}}\int_0^t \frac{z\cos 2\overline{\omega}\zeta}{\zeta\sqrt{\zeta}} e^{-\frac{qz^2}{4\mu\zeta}} d\zeta.$$

As

$$\int_\varepsilon^t \left| \frac{z}{\zeta\sqrt{\zeta}} e^{-\frac{qz^2}{4\mu\zeta}} \right| d\zeta = 0,$$

if $\varepsilon > 0$ and $z=0$, the value of the above expression for $\frac{\partial v}{\partial z}$ is in the case of $z=0$ not altered, if the factor $\cos 2\overline{\omega}\zeta$ be replaced in it by 1. With the substitution $\zeta = xz^2$ it consequently gives

$$\left(\frac{\partial v}{\partial z}\right)_{z=0} = -\frac{T\sqrt{q}}{2\mu\sqrt{\mu\pi}}\int_0^\infty \frac{e^{-\frac{q}{4\mu x}}}{x\sqrt{x}} dx = -\frac{T}{\mu}.$$

Equations (10) thus give the solution of the given problem.

[1] This name was kindly suggested by Prof. H. GEELMUYDEN.

various depths, *i. e.* the curve described by the point of an arrow issuing from a fixed point and representing in magnitude and direction the actual velocity of the water at every instant. Figs. 3—6 represent the hodographs, constructed by

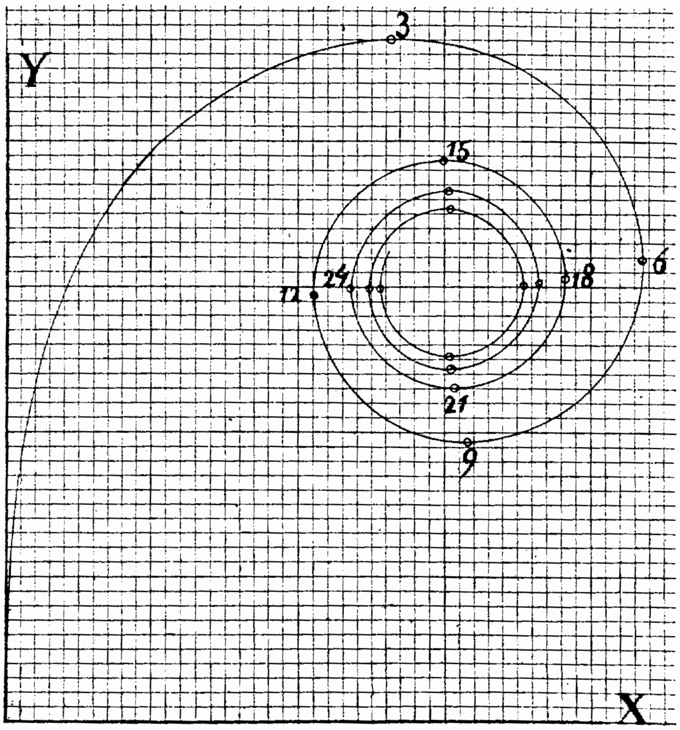


Fig. 3. $z = 0$.

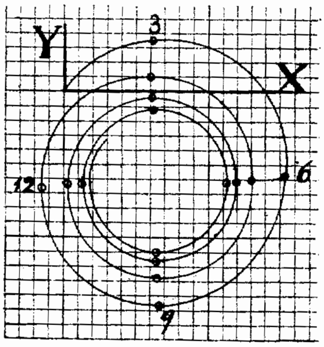


Fig. 4. $z = 0{,}5\,D$.

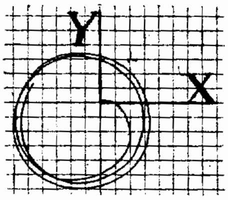


Fig. 5. $z = D$.

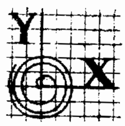


Fig. 6. $z = 2D$.

means of equations (11) at the surface and at depths $z = 0{,}5\,D$, D, and $2\,D$. The unit of length is chosen arbitrarily but equal in the case of all the figures. The arrows denoting velocity are not drawn, but their end-points after 3, 6, 9, 12, 15 etc. pendulum-hours, are denoted by a small circle and the corresponding number. It is seen that the hodographs are spirals, every 12 pendulum-hours describing one turn round the point corresponding to the stationary state of motion. (The *path* described by a water-particle is of course quite different). *The average velocity over 24 pendulum-hours, is thus from the very beginning practically stationary.* In a sea of small depth the stationary state of motion will of course be established still more quickly than in one of infinite depth.

On the other hand the periodical deviations from the average velocity at a given depth, abate only slowly (inversely as the square root of the time); as the period of these deviations is in high latitudes approximately the same as that of the tidal currents, it might be necessary when studying the latter, to take account of the errors which might possibly be introduced as a result of interference between the tidal currents and the wind-currents.

Neither do the actual velocities decrease with the depth as rapidly as do the average velocities represented in Fig. 1. Thus for instance, if the final velocity at the surface be 1, the velocity at the depth D reaches after 24 pendulum-hours the maximum value $^1/_7$ *i. e.* more than three times the average velocity at the same depth; and at the depth $2\,D$ it reaches after about 60 pendulum-hours the maximum value $^1/_{20}$ or 27 times the average velocity at the same depth. These periodical deviations from the average velocities have moreover, at any particular moment the same direction at all depths; which circumstance will considerably increase their importance.

II. Currents caused by pressure-gradient and the earth's rotation alone.

The immediate effect of the continents upon a wind-current, is to check the motion of the water in certain directions, so that it will be stored up against or sucked out from the coast, as the case may be. The inclination of surface thus arising will gradually increase until the current caused by it is sufficient to restore the equilibrium between inflow and outflow from the coast. In order to find the influence of the continents on the ocean-currents the first step must therefore be the calculation of the currents produced by a constant inclination of surface. This is equivalent to a uniform horizontal pressure-gradient.

Assume as before, an infinite ocean of uniform depth d and uniform density q. The axis of x and y may be laid on the sea-surface, the latter 90° to the left of the former (when the northern hemisphere is considered), and the axis of z perpendicular to them and reckoned positive downwards. Further suppose the surface to be inclined at a constant angle γ in such a direction as to make

$$X = 0; \quad Y = g \sin \gamma. \quad (g = \text{gravity}).$$

The equations of steady motion are then, the same as (3) p. 6 with the addition only of a term depending on the gravity; or:

$$(12) \qquad \frac{d^2u}{dz^2} + 2a^2v = 0; \quad \frac{d^2v}{dz^2} - 2a^2u + \frac{qg \sin \gamma}{\mu} = 0.$$

The integrals of these equations are

$$(13) \qquad \begin{aligned} u &= C_1 e^{az} \cos (az + c_1) + C_2 e^{-az} \cos (az + c_2) + \frac{qg \sin \gamma}{2a^2\mu} \\ v &= C_1 e^{az} \sin (az + c_1) - C_2 e^{-az} \sin (az + c_2). \end{aligned}$$

The supposition of no wind on the surface implies

$$\frac{du}{dz} = \frac{dv}{dz} = 0 \text{ for } z = 0,$$

which gives

$$C_1 = C_2 = \frac{1}{2} C; \; c_1 = -c_2 = c,$$

C and c being new arbitrary constants. If further, the value $q\omega \sin \varphi$ be substituted for $a^2\mu$, Equations (13) may be written in the form

$$(14) \qquad \begin{aligned} u &= C [\operatorname{Cosh} az \cos az \cos c - \operatorname{Sinh} az \sin az \sin c] + \frac{g \sin \gamma}{2 \omega \sin \varphi} \\ v &= C [\operatorname{Cosh} az \cos az \sin c + \operatorname{Sinh} az \sin az \cos c]. \end{aligned}$$

At the bottom ($z = d$), $u = v = 0$; and consequently

$$C \sin c = \frac{g \sin \gamma}{\omega \sin \varphi} \cdot \frac{\operatorname{Sinh} ad \sin ad}{\operatorname{Cosh} 2\,ad + \cos 2\,ad}$$

$$C \cos c = -\frac{g \sin \gamma}{\omega \sin \varphi} \cdot \frac{\operatorname{Cosh} ad \cos ad}{\operatorname{Cosh} 2\,ad + \cos 2\,ad}.$$

With these values Equations (14) finally take the form

$$(15)\quad \begin{cases} u = -\dfrac{g \sin \gamma}{2\,\omega \sin \varphi} \dfrac{\operatorname{Cosh} a(d+z) \cos a(d-z) + \operatorname{Cosh} a(d-z) \cos a(d+z)}{\operatorname{Cosh} 2\,ad + \cos 2\,ad} + \dfrac{g \sin \gamma}{2\,\omega \sin \varphi} \\ v = +\dfrac{g \sin \gamma}{2\,\omega \sin \varphi} \dfrac{\operatorname{Sinh} a(d+z) \sin a(d-z) + \operatorname{Sinh} a(d-z) \sin a(d+z)}{\operatorname{Cosh} 2\,ad + \cos 2\,ad}. \end{cases}$$

The curves Fig. 7 are calculated from these equations and represent quite in the same way as in the case of Fig. 2,

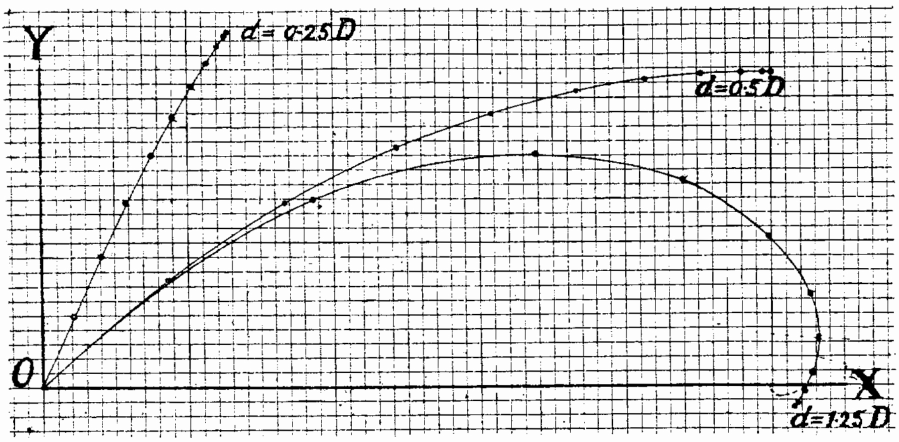


Fig. 7.

the velocity at different levels above the sea-bottom. The depth of the water is 0,25 D, 0,5 D, 1,25 D and for the dotted continuation of the last curve, 2,5 D respectively; the points on each curve denote the end-points of the arrows representing the velocities at 0,1, 0,2 etc. of this distance from the bottom. The greater the depth of the water, the more is the current deviated from the direction of the pressure-gradient. If the depth be sufficiently great ($> D$) the current consists of *a bottom-current* of thickness D running more or less in the direction of the force, and above this *a current reaching right up to the surface with almost uniform velocity*

$$(16)\qquad u_0 = \frac{g \sin \gamma}{2\,\omega \sin \varphi}; \quad v_0 = 0$$

perpendicularly to the force. The sea-bottom obviously acts as a wind on the under side of the water as it runs with uniform velocity over it, and in this way produces the outgoing current. In the case of infinite depth the curve representing the gradient-current is exactly similar to the curve in Fig. 1 representing the wind-current, except that the arrows issue from the opposite end of the curve; immediately above the bottom the direction of current forms an angle of 45° with the direction of force[1].

The total flow of water in the directions of x and y respectively is, owing to (12)

$$S_x = \int_0^d u dz = \left[\frac{1}{2a^2} \frac{dv}{dz} + \frac{qgz \sin\gamma}{2a^2\mu} \right]_{z=0}^{z=d}$$

$$S_y = \int_0^d v dz = \left[-\frac{1}{2a^2} \frac{du}{dz} \right]_{z=0}^{z=d}$$

and from these equations and Equations (15) it follows, on substituting for a^2 its value $q\omega \sin\varphi/\mu$, that

$$(17) \quad \begin{aligned} S_x &= \frac{Dg \sin\gamma}{4\pi\omega \sin\varphi} \left(2ad - \frac{\operatorname{Sinh} 2ad + \sin 2ad}{\operatorname{Cosh} 2ad + \cos 2ad} \right) \\ S_y &= \frac{Dg \sin\gamma}{4\pi\omega \sin\varphi} \cdot \frac{\operatorname{Sinh} 2ad - \sin 2ad}{\operatorname{Cosh} 2ad + \cos 2ad} . \end{aligned}$$

It is seen from the second of these equations, that the total flow in the direction of the pressure-gradient does not increase indefinitely with the depth of the ocean but approximates to the limit $Dg \sin\gamma/4\pi\omega \sin\varphi$.

The current in its different stages of development remains to be considered. To begin with, let us suppose that the water initially at rest, suddenly at the time $t=0$ becomes subjected to a uniform and constant force directed along the axis

[1] The present problem may also be applied to the case of the atmosphere as far as the isobaric surfaces concerned can be regarded as equidistant and parallel. It agrees with the known fact that the wind direction in the northern hemisphere lies about 45° to the right (in the southern hemisphere to the left) of the direction of the gradient, the angle between them further increasing from the earth's surface upwards. This remark must at present be made with reservation, since the *acceleration* of the air may be of considerable importance as a result of the large velocities and curved paths of the winds.

of positive y, and equivalent to an inclination γ of the surface. The equations of motion are then

$$\frac{\partial u}{\partial t} = 2\,\bar{\omega} v + \frac{\mu}{q}\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial v}{\partial t} = -\,2\,\bar{\omega} u + g \sin\gamma + \frac{\mu}{q}\frac{\partial^2 v}{\partial z^2}$$

where as before, $\bar{\omega} = \omega \sin\varphi$.

If there were no friction at all ($\mu = 0$) these equations would give

$$(18) \qquad u = \frac{g \sin\gamma}{2\,\bar{\omega}} - \frac{g \sin\gamma}{2\,\bar{\omega}} \cos 2\,\bar{\omega} t\,;\; v = \frac{g \sin\gamma}{2\,\bar{\omega}} \sin 2\,\bar{\omega} t\,,$$

and the water-particles would consequently move along cycloids

$$x = x_0 + \frac{g \sin\gamma}{4\,\bar{\omega}^2}[2\,\bar{\omega} t - \sin 2\,\bar{\omega} t]$$

$$y = y_0 + \frac{g \sin\gamma}{4\,\bar{\omega}^2}[1 - \cos 2\,\bar{\omega} t].$$

The velocity (18) may be divided into two parts: one uniform velocity u_1 perpendicular to the force, and one variable velocity (u_2, v_2) producing a circular motion. On account of the friction against the bottom the complete solution of our problem will contain two more terms. The first one represents the motion which would be produced in water initially at rest if the bottom suddenly began to move with the uniform velocity $u = -u_1$. This is in the case of infinite depth analogous to (10) and may be found from it by differentiation with regard to z. But it converges much more rapidly than the integral (10), and gives together with u_1 almost immediately, the steady motion (15) represented in Fig. 7. The other supplementary term represents the motion which would be produced in water initially at rest, if the bottom suddenly began to move with the variable velocity $u = -u_2$; $v = -v_2$. As is easily seen it may be expressed in the case of d infinite by

$$u = -\,u_2\left[1 - P\frac{z\sqrt{\pi}}{2\,D\sqrt{\tau}}\right];\; v = -\,v_2\left[1 - P\frac{z\sqrt{\pi}}{2\,D\sqrt{\tau}}\right],$$

where z is the distance from the bottom, $\tau = t\frac{\overline{\omega}}{\pi}$ is the time expressed in units of 12 pendulum-hours, and Px is the Probability Function[1]

$$Px = \frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}dx\,.$$

It represents the extinction by viscosity of the circular motion $u = u_2$; $v = v_2$. This extinction takes a considerable time; in the present case of infinite depth it would for instance last 306 pendulum-days each of 24 pendulum-hours, before the amplitude of the circular motion at a height $5\,D$ above the sea-bottom, would have decreased to a fifth. At other levels this time would vary as the square of the height above the sea-bottom; and furthermore the residue of motion decreases approximately as the reciprocal of the time.

When the inclination of the sea-surface is produced by the wind, it will increase more or less *gradually*, and the periodical motion will then of course, be less appreciable. If for example $\sin\gamma$ increases uniformly with the time ($g\sin\gamma = kt$, where k is a constant) the equations of motion are, in the absence of friction

$$\frac{du}{dt} = 2\,\overline{\omega}v;\ \frac{dv}{dt} = -2\,\overline{\omega}u + kt\,.$$

If we assume $u = v = 0$ initially, they give

$$u = \frac{k}{4\,\overline{\omega}^2}(2\,\overline{\omega}t - \sin 2\,\overline{\omega}t)$$

$$v = \frac{k}{4\,\overline{\omega}^2}(1 - \cos 2\,\overline{\omega}t),$$

i. e. the *hodograph* of the motion is in this case a cycloid situated quite similarly to the cycloidal paths of the water particles in the case treated above, one branch of the cycloid being described every 12 pendulum-hours. When the current has been on the increase for two or three days or more, the periodic inequalities in the velocity are therefore practically insensible compared with the total velocity, which is nearly

[1] Tabulated by ENCKE in *Berliner Astronomisches Jahrbuch* 1834 p. 305 ff.

perpendicular to the gradient. The total motion of the water in the direction of the force is, if the periodical oscillations be disregarded,

$$\frac{kt}{4\,\overline{\omega}^2} = \frac{g \sin \gamma_0}{4\,\overline{\omega}^2} \tag{19}$$

γ_0 being the final inclination of the surface.

In summary, the effect of a horizontal pressure-gradient uniform from the bottom to the surface over a large portion of the sea, is 1:) a steady current of the kind represented in Fig. 7, practically following the variations of strength and direction of the force, instantaneously, 2:) an oscillation of the whole bulk of water in circular paths, which becomes extinguished from the bottom upwards. The extinction goes on very slowly — during months say — in the case of a deep ocean. But on the other hand the oscillatory motion will from the beginning be inconsiderable compared with the steady motion, if the rise or change of the pressure-gradient has been proceeding uniformly for two or three pendulum-days or more. And this is, as will be shown below, always the case in the open ocean.

The significance of the above result may be made somewhat clearer by help of a simple application. Imagine a sea having the shape of a square-sectioned channel. If there be a continual supply of water at the one end of the channel, the primary result will be an inclination of surface *along* the channel and a stream of water in the same direction. Owing to the earth's rotation the current will be deviated to the right (in the northern hemisphere) until a certain inclination of the surface *transversely* to the channel has arisen and restored the equilibrium, *i. e.* so as to make the total flow transversely to the direction of the channel *nil*. The total inclination will be directed approximately *along* the channel if d/D is small; but nearly perpendicularly to it if d is greater than D, and in this case the original slope will be only a small down-stream component of the final inclination. The motion consists of a current in the direction of the channel and of a circulation in the planes perpendicular to this direction. This circulation consists, if the depth d is much greater than D, of a bottom current running to the left up to the level D about, and between this level and the surface a slow and nearly uniform current in the opposite direction.

The solutions (8) p. 13 and (15) p. 21 when duly combined, also represent the stationary motion in the case of any number of superposed water-layers of different but uniform densities. The boundary-conditions which are to be fulfilled would however as a rule be very complicated and the method therefore of little practical value; and we may content ourselves with the suggestion afforded by the fact that the general solution of the problem must be made up of solutions of the forms (8) and (15).

On the other hand a characteristic case in which the density increases uniformly downwards, is very easily treated. Together with the problems treated above it will give a sufficiently good idea of the part taken by the earth's rotation in *convection currents.* We will assume that the surfaces of equal density are planes parallel to one another, and situated so that

$$\frac{\partial p}{\partial x} = 0; \quad -\frac{\partial p}{\partial y} = b(d-z)$$

d being the depth at which the isobaric surface is *horizontal.* Further we may assume for the present, that the velocity is *nil* at the same depth d. The equations for stationary motion become

$$\frac{d^2u}{dz^2} + 2a^2v = 0; \quad \frac{d^2v}{dz^2} - 2a^2u + \frac{b}{\mu}(d-z) = 0,$$

and with the conditions

$$\frac{du}{dz} = \frac{dv}{dz} = 0 \text{ for } z = 0; \quad u = v = 0 \text{ for } z = d,$$

they give

$$(20)\begin{cases} u = \dfrac{bd}{4\pi q\omega \sin\varphi} \cdot \dfrac{D}{d}\Big[A \operatorname{Cosh} az \cos az \\ \qquad + B \operatorname{Sinh} az \sin az - e^{-az}(\cos az - \sin az) + \dfrac{2\pi(d-z)}{D}\Big] \\ v = \dfrac{bd}{4\pi q\omega \sin\varphi} \cdot \dfrac{D}{d}\Big[A \operatorname{Sinh} az \sin az \\ \qquad - B \operatorname{Cosh} az \cos az + e^{-az}(\cos az + \sin az)\Big] \\ A = 1 - \dfrac{\operatorname{Sinh} 2ad + \sin 2ad}{\operatorname{Cosh} 2ad + \cos 2ad}; \quad B = 1 - \dfrac{\operatorname{Sinh} 2ad - \sin 2ad}{\operatorname{Cosh} 2ad + \cos 2ad}. \end{cases}$$

The result is represented in Fig. 8 by curves drawn for the cases of $d/D = 0{,}25$, $0{,}5$, $1{,}25$ and $2{,}5$. The coefficient $bd/4\pi q\omega \sin\varphi$ is assumed $= 1$ in the case of all the curves, and the signification of them is in other respects the same as in Figs. 2 and 7.

If the variation of the pressure-gradient followed exactly the same law down to the depth $2d$ (being there equal but of opposite sign to the gradient in the surface), and if further the water moved without friction at the depth $2d$, the same equations (20) would hold unaltered. In this way they would represent a simplified case of convection currents in the uppermost water-strata of the sea. If the assumption ($u = v = 0$ for $z = d$) is not true, we have to add to (20) solutions of the form (8), (15) and (16) so as to satisfy the

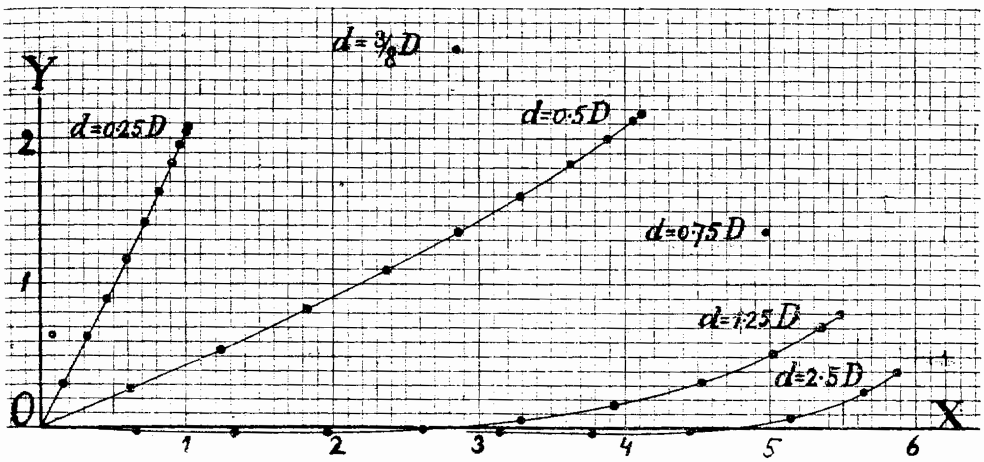


Fig. 8.

boundary-conditions within the separate water-strata. There is no particular difficulty in solving in a similar way, a whole series of problems corresponding tolerably well to different conditions actually occurring in the sea.

Prof. V. BJERKNES has given an extremely clear and simple formula representing the general laws of motion of the sea (and atmosphere) on the rotating earth[1]. Consider any closed curve in the sea (or atmosphere) always bound to the material particles through which it originally passed. S may be the area enclosed by the curve's projection on the equatorial plane and A the number of »*solenoids*» enclosed by the curve, *i. e.* the number of tubes formed by

[1] V. Bjerknes l. c. p. 1.

intersection between *isobaric* and *isosteric* surfaces, the pressure and specific volume increasing by unity from surface to surface. Finally C may be the circulation of the curve (according to Lord KELVIN's definition). C as well as S and A have to be formed with due consideration paid to the sign. V. BJERKNES' theorem gives then, the rate of increase of circulation with time, in the form

$$\frac{dC}{dt} = A - R - 2\,\omega \frac{dS}{dt} \tag{21}$$

where $2\,\omega dS/dt$ expresses the influence of the earth's rotation. and R is a symbol of the effect of friction. A great difficulty which has to be overcome if this formula should be made as practicable as possible, is that the analytical expression of R is very inconvenient for numerical calculation, even if the coefficient of friction of the fluid be known.

In the particular case of steady motion represented by (20) this difficulty is avoided, since the complete solution of the problem is contained in the curves in Fig. 8. It is seen that the direction of velocity is in all depths nearly parallel to *one* direction; and this forms a greater angle with the direction of the force, the greater the ratio d/D; d corresponding to about half the height of the layer within which the circulation takes place. The result of the investigation pp. 22—25 indicates without a doubt that convection-currents as well, may as a rule be regarded as stationary.

Even in the more general case where the »solenoids» are not uniformly distributed but more or less concentrated to thinner transition-layers, the curves in Fig. 8 may give some indication as to the *direction* of the surface-current. As a matter of fact this direction is for a given value of d/D, nearly the same in Fig. 7 and Fig. 8, though the former represents the extreme case of convection-currents in which all solenoids are concentrated to a single plane at the depth d in the middle of the considered water-layer; and it seems therefore as if a different distribution of the solenoids does not affect very much the angle between the surface-velocity and the direction of the force. The curves in Figs. 7 and 8 may thus be of help in discussing the convection currents by means of BJERKNES' formula (21). If for instance the thickness $2\,d$ of the layer in which the convection takes place, is $0{,}5\,D$, the friction will be of greater importance than the earth's rotation, and the currents when stationary will be deflected about 30° from the direction of the force; if $2d = D$ the friction is of inferior importance and the stationary currents will be deflected about 60° from the force-direction; if $2d$ is greater than $5D$ the friction may almost entirely be disregarded. The absolute difference of velocity in the surface and at the depth $2d$ may in either case be roughly estimated by applying (21) to a curve laid perpendicularly to the direction of this velocity-difference; since R then becomes comparatively unimportant.

It is obvious that a point to be especially examined is to what extent the quantity D may be different in the case of homogeneous water and in water of density gradually increasing downwards.

III. Wind-currents influenced by the continents etc.

In section I the laws of wind-currents in an infinite ocean have been studied under the supposition that the water is influenced by no other forces than the tangential pressure exerted by the wind itself, and the »deflecting force» due to the earth's rotation. Actually however the currents will be more or less influenced by differences of density in the water and by the resistance against the motion of the water due to land or to water outside the region considered. The boundary conditions which are to be satisfied in addition to the equations of motion may therefore be more or less complicated. In the case of steady motion they will always include the stipulation that the quantity of water flowing into any region considered must be equal to the quantity flowing out of it — otherwise the water-surface within that region would be steadily rising or steadily sinking.

The stationary motion will obviously be established in such a way that the water-level is raised or lowered in various places, and particularly along the coasts (»*Wind-stau*»), until the pressure-gradient thus arisen restores the equilibrium between the flow to and from any part of the ocean. In case the sea is made up of water-layers of markedly different specific gravities the equilibrium must be established within each water-layer separately, if the motion shall be strictly stationary. For the present, only the case of homogeneous water will be examined. The motion is then, when stationary, made up of a pure wind-current as studied in section I and a current driven by a uniform pressure-gradient which is equal from the surface right down to the bottom. The disturbing effect of the continents in forcing the water-particles away from their *horizontal* paths, and the friction between water-particles in the same horizontal plane, will certainly be quite insignificant except in the very neighbourhood of the coast; since even at a comparatively moderate distance from the latter, the vertical distances are many times smaller than the horizontal ones.

Problem a. Case of a long straight coast.

Suppose a steady uniform wind blowing in a constant direction everywhere outside a straight[1] and infinitely long coast. The sea may as before be of uniform depth. Since the current must then be alike in any two places at the same distance from the coast, no inclination of the surface can occur in the direction of the coast itself; but perpendicular to this direction a slope will arise and gradually increase until the total flow — due to wind and pressure-gradient — perpendicularly to the coast, is *nil*.

The equations for a steady current will then be obtained by adding (8) and a solution of the form (15), the direction of y being in the former that of the wind, and in the latter perpendicular to that of the coast; the coefficient $\sin\gamma$ in (15), to realize the condition $S=0$ perpendicularly to the coast, has to be determined by help of (9) and (17). The

[1] The stipulation that the coast should be straight, may as a rule be dispensed with, provided that the direction of the wind *relative to the coast line* is the same everywhere.

A current of velocity V which is influenced by *no* forces, would under the influence of the earth's rotation move in a circular path of radius $r = V/2\,\omega\,\sin\varphi$. On the latitude of the Bay of Biscay ($\varphi = 45°$) this radius would be 10 km only, and in Skagerak 8 km, even if we assume $V = 1$ m. per sec., which indeed may be regarded as an extreme limit for the velocity. On the other hand the outline of the Bay of Biscay has a radius of more than 200 km and the coast north of Skagerak ca 60 km. Even in the case of such sharp bends as these two, the forces arising as a result of the coast's curvature will therefore be insignificant compared to the forces due to the earth's rotation alone.

character of the motion will of course depend on the depth of the sea *d* compared to the Depth of Wind-current *D*, and on the angle between the wind direction and the coast-line. This dependence may be seen from Figs. 1—15, Pl. I — representing in quite the same way as Figs. 2, 7 and 8, the velocity of the current in different levels. The cordinate-axis are not drawn, but the origin is represented by a circle at the end of each curve. The scale of velocities is the same in the case of all the figures and the direction and velocity of the wind is represented by an arrow at the lower left corner of the plate; the direction of the coast is indicated for each horizontal group of 4 figures and the depth *d* for each figure separately. If the depth is sufficiently great, there will be three currents each quite distinct from one another, namely 1:) a *bottom-current* of depth *D* moving more or less in the direction of the slope, though with a deflection to the right increasing from 45° at the bottom to 90° at the top. 2:) a »*midwater-current*» of almost uniform velocity parallel to the coast and reaching from the top of the bottom-current up to the depth *D* below the surface. It is represented by a group of points close to one another in the middle of the curve (see Figs. 1, 5, 9). 3:) above this a *surface-current* in which the velocities are equal to those of a wind-current superposed on the velocity of the midwater-current. The bottom-current and the surface-current will not be appreciably influenced by an alteration of the depth *d*, as long as this exceeds 2 *D*, and the only effect then will be a corresponding alteration of the depth of the uniform midwater-current. (In the case of Figs. 1, 5, 9, there is only the beginning of a midwater-current since the depth $^5/_2$ *D*, does not exceed 2 *D* by very much). If the depth is smaller than 2 *D*, the midwater-current fails and the bottom-current and surface-current pass evenly into one another, with the loss, more or less, of their characteristic form (Figs. 2—4, 6—8, 10—12). Figs. 13—15 illustrate the case in which the wind blows perpendicularly to the coast. In this case the total flow in the primary wind current is directed parallel to the coast, if d/D is not small; and no bottom-current and midwater-current are therefore created.

The most striking result of the coast's influence is that *a wind is able indirectly to produce a current more or less in* *its own direction from the surface down to the bottom,* while in the absence of coasts the wind's effect would be limited to a comparatively thin surface-layer. *The bulk of this current — the »midwater-current» — is directed along the coast and its velocity is proportional simply to the wind component parallel to the coast.* The rate and direction of the surface current, is also largely influenced by the direction of the coast. As would be expected, a wind directed *along* the coast produces a much greater velocity than would a wind blowing perpendicularly to this direction. Furthermore the wind's direction to the *left* or to the *right* of the coast-line is — as will be easily seen from Figs. 1—4 and 9—12 — of great importance except when the depth d is considerably smaller than D, in which case the earth's rotation is of little influence. The relationship between the relative direction of wind and coast-line on the one hand, and the velocity and direction of motion of the surface-water on the other, is very simple in the case of d infinite. The end-points of the arrows denoting surface-velocity are situated on a circle of radius $r = V_0/\sqrt{2}$ (equation 5, p. 7) and with centre at the point $x = V_0/\sqrt{2}$; $y = 2\, V_0/\sqrt{2}$, the axis of y being in the wind's direction (see Fig. 16 Pl. I). If furthermore AD in the same figure be the direction of the coast, the surface-velocity is represented by the line OD[1].

It follows from this construction that the surface-current is always deflected to the right of the wind's direction; the angle of deflection is between 0 and 53° and its mean value is 26°,5. A wind of given strength would have the greatest effect when directed a little more than 13° to the left of the coast-line; and perpendicularly to this direction it would have its smallest effect, the velocity of the surface-water in the two cases being in the ratio 8 to 3 about. If

[1] This is easily proved. If β be the acute angle between the coast line and the wind's direction, taken positively to the right of the latter, the total flow of water perpendicularly to the coast produced by the wind current alone, is according to (7) $(V_0 D/\pi\sqrt{2}) \cos \beta$. This must then be equal to the flow S_y (Equation 17), and it follows from this and (16) that the velocity of the midwater-current is $U_0 = V_0\sqrt{2} \cos \beta$ along that direction of the coast which forms an acute angle with the wind-direction. If the angle BAD be β, $AD = AB \cos \beta$ then represents in magnitude and direction the midwater-current; OA is the surface velocity due to the wind alone, and the resultant velocity of the surface-water is consequently OD.

the wind be conceived as having all possible directions relative to the coast, the average velocity of the surface water would be in the ratio 1,66 to the value it would have in the absence of coasts.

It may be mentioned that with a given wind, the velocity of the midwater-current depends only on the value that D has within the region of the *bottom-current*. For as the total in- or out-flow produced by the surface-current depends only upon the wind-pressure T itself and not upon the value of D (see p. 9—10) it is quite the same in the case of the bottom-current. It follows then from (16) and (17) that the velocity of the midwater-current with a given wind will vary inversely as the value of D approximately, within the region of the bottom-current. If this be greater than at the surface the midwater-current will be comparatively slower than here calculated, and *vice versa;* it is difficult however, for the present to say which of these two alternatives is the more likely to be the true one.

An estimation of the time required for the stationary motion to become established, cannot be made with the same degree of accuracy as in the preceding simpler cases, because in the present case the quantity D enters more essentially into the result. We must therefore be satisfied with results based upon a hypothetical value of D.

It has been shown above that the steady currents corresponding to given degrees of inclination of the surface arise practically simultaneously with the inclination itself — at all events if we make allowance for periodical deviations from the average velocities. We have therefore to calculate only the time required for the wind-current (or its excess over the rising bottom-current) to carry sufficient water towards or from land 1:) to produce the inclination of surface corresponding to the stationary state of motion; 2:) to compensate for the outflow (or inflow) of water accompanying the rise of the midwater-current.

1:) Let $z = f(x, t)$ be the equation of the sea-surface, z being the height of the sea-level, x the distance from the coast, and t the time. The actual inclination of the surface at any instant may be γ, and its final value γ_0. Since γ is always very small, it is

$$(22) \qquad \gamma = \frac{\partial z}{\partial x}.$$

If S_0 be the flow of water (per unit length of the coast) in the direction of x, due to the wind-current alone and S or $S(t)$ the total flow in the same direction, due to wind-current and bottom-current, we may assume as a first approximation

$$(23) \qquad S = S_0 \frac{\gamma_0 - \gamma}{\gamma_0}.$$

Further it is obvious that

$$-\frac{\partial S}{\partial x} = \frac{\partial z}{\partial t};$$

which as a result of the equality (23) takes the form

$$\frac{S_0}{\gamma_0} \frac{\partial \gamma}{\partial x} = \frac{\partial z}{\partial t}$$

and then on account of (22)

$$\frac{S_0}{\gamma_0} \frac{\partial^2 \gamma}{\partial x^2} = \frac{\partial \gamma}{\partial t}.$$

Further $\gamma = 0$ for $t = 0$ and $\gamma = \gamma_0$ for $x = 0$ (at the coast). If the ocean's extension in the direction of x be regarded as infinite, the determination of γ is then from an analytical point of view, a problem well known in mathematical physics. It is

$$(24) \qquad \gamma = \gamma_0 \left[1 - P\left(\sqrt{\frac{\gamma_0 x^2}{4 S_0 t}} \right) \right],$$

where P is the function of probability (see p. 24). The second of Equations (17) p. 22 gives approximately

$$S_0 = \frac{D g \gamma_0}{4 \pi \omega \sin \varphi}.$$

For values of y less than 0,4

$$P(y) = 1{,}1\, y \text{ roughly},$$

and (24) then gives

$$(25) \qquad \frac{\gamma_0 - \gamma}{\gamma_0} = 1{,}1 \sqrt{\frac{\pi \omega \sin \varphi}{D g} \cdot \frac{x^2}{t}}.$$

If we assume for example $\varphi = 45^\circ$ we have in the meter and second units

$$\frac{\pi\omega \sin \varphi}{g} = 0{,}0000165\,,$$

so that the time in which the inclination of the surface would have attained say 0,7 of its final value, is

$$t_1 = 0{,}00022\; x^2/D. \tag{26}$$

To get at least an idea of the times which would come into question we may assume for the moment, $D = 75$ m. This assumption gives the times

$t_1 = 3$	seconds	at 1000	m.	distance	from	the	coast
$t_1 = 5$	minutes	10	km.	»	»	»	»
$t_1 = 8$	hours	100	»	»	»	»	»
$t_1 = 8\,^1/_2$	days	500	»	»	»	»	»
$t_1 = 34$	»	1000	»	»	»	»	»

At latitudes other than 45° these times have to be altered proportionally to sin φ.

Actually the inclination should be formed more rapidly than would follow from the above calculation (perhaps in half the time) over the whole region of the wind-current, since the water which is mounted up towards the coast is taken from the outer part of the region and leaves a hollow there. A more accurate calculation taking this circumstance into account would, even if possible, be of little value, because for other reasons we cannot get results correct enough to give more than the order of magnitude. If the value of D assumed, be too small, the calculated values of t will be proportionately too great and *vice versa*.

2:) There remains to be calculated the time required for the second purpose. According to what was said on p. 23—25 the midwater-current cannot attain its velocity perpendicularly to the slope without first having moved a corresponding distance in the direction of the slope. The quantity of water which must thus be carried by the rising midwater-current through any vertical plane parallel to the coast, is approximately proportional to the depth of the sea d. According to (19) it is $gd \sin \gamma / 4\,\overline{\omega}^2$ per unit of length of the coast. The same quantity of water has then to be carried in the opposite direction by the wind-current. If we assume that in the time considered, *half* the flow of the wind-current is neutralized by the bottom-current, the rate of transport would be $\frac{1}{2} S_y$ (Equation 17), or approximately

$$Dg \sin \gamma / 8 \pi \overline{\omega}.$$

The time required would then be

$$\frac{2 \pi d}{\overline{\omega} D} \text{ seconds}$$

or

$$t_2 = \frac{d}{D} \text{ pendulum-days,} \tag{27}$$

which time has to be added to the time t_1 found on p. 35. If the assumption $D = 75$ m. be not too erroneous, we may conclude from (26) and (27) that the stationary state of motion will be practically established to within some hundred km. from land in a few days, if the depth does not exceed 200—400 m. say (*i. e.* on the continental shelves). In the deep ocean on the other hand, particularly in the case of very broad currents (1000 or 2000 km. say), the midwater-current may require several months to become approximately fully developed. From this calculation it would seem as though the midwater-currents are, as a rule, to some extent able to follow the changes of the monsoon-winds. The surface-current changes with the varying winds even within the first 12 pendulum-hours (p. 17 seq.), and it is very probable that the periods of monsoon drifts are to a large extent chiefly due to the surface-current.

SANDSTRÖM has pointed out[1] that simply by observations of water-level at both sides of a channel — as for instance on the coasts of Iceland and Norway — it is possible to calculate the average velocity of the current through the channel; and he has also given convenient tables for using this method in the case of stationary currents, as far as the effect of friction can be disregarded. It may be easily seen that as far as concerns wind-currents, SANDSTRÖM's method would as a matter of fact give the *midwater-current*, and it might with due precautions give very valuable information

[1] J. W. SANDSTRÖM: Ueber die Anwendung von Pegelbeobachtungen zur Berechnung der Geschwindigkeit der Meeresströme. Svenska Hydrografisk biologiska kommissionens skrifter. I. Göteborg 1903.

in this connection. And especially it might be possible in this way to find the extent to which the midwater-current alters from one season to another.

Problem b. Case of an enclosed sea.

Consider the case of a sea enclosed on all sides by land, and impelled by a wind blowing everywhere with the same strength and direction. In this case the total flow in *any* direction must be *nil* when steady motion shall have been established, and the direction and magnitude of the slope for any wind is then easily calculated by means of equations (9) and (17). Just as in the case *a* it is possible if d is greater than $2\,D$, to distinguish clearly a bottom-current of depth D, a surface-current of the same depth, and between the two a »midwater-current» of uniform velocity. This latter velocity will however be much smaller than in the case of the wind blowing along a straight coast, and furthermore it is smaller the greater the depth d — roughly inversely proportional to the latter. The surface current is consequently only slightly influenced by the coast, if the depth is great. Figs. 17—19 Pl. I represent in the same manner as Figs. 1—15 the current in the cases of $d = 2{,}5\,D$, $d = 1{,}25\,D$ and $d = 0{,}5\,D$. The arrows represented without shaft-feathers give the direction of the slope; it is remarkable how nearly this direction follows the wind's direction (common for the whole plate) whatever be the depth of water. This shows clearly that *the earth's rotation has no considerable deflecting influence on the mounting up of water, in a sea impelled over its whole area by the same wind* (although the currents themselves may deviate from the wind's direction). *Its influence on the absolute magnitude of the mounting up is also found to be rather moderate,* its effect being to diminish the inclination of the water-surface in the ratio $0{,}98$ if $d = 0{,}5\,D$, in the ratio $0{,}77$ if $d = 1{,}25\,D$, $0{,}71$ if $d = 2{,}5\,D$, and exactly $^2/_3$ if d is infinite.

The stationary motion in developing obviously passes through two stages, though of course with an even transition from the one stage to the other. The current deviates first towards the coast on the right, until the inclination of surface corresponding to a current along a straight coast, has arisen. The »midwater-current» caused by this inclination then stores up the water in the direction of the wind until the final state is reached. These results are confirmed by observations from the Baltic collected by COLDING[1] after the great Storm in November 1872. On this occasion the lines of equal water-level stood nearly perpendicular to the wind's direction except during the first day the storm was blowing; during this day they had a direction markedly more to the right.

As a rule Problem *a* will not exactly represent actual cases, and because neither the continent nor the region of uniform wind can be regarded as of infinite extension. If the midwater-current has to overcome the resistance offered to it by another current or by a dead mass of water at the boundary of the region considered, a small inclination of surface in the direction of the midwater-current is produced. The velocity of the latter and the slope perpendicularly to it, are consequently diminished, and the result will be something between that of Problem *a* and of Problem *b*. The difference of level thus created between the front and back of the current, has to produce a compensation-current completing the circulation of the water in another part of the ocean. Similarly the midwater-current (and with it the surface-current) may be *increased* by the effect of neighbouring currents. The latter case will however be less common than the former (it would be of equally frequent occurence if the wind-currents always formed complete circulations in cyclonic or anticyclonic directions) and the velocities of wind-currents will therefore upon an average be somewhat slower than calculated in Problem *a*. With this restriction Problem *a* may with due precautions obviously be applied also to currents running alongside one another in opposite directions or with different velocities, in the open ocean, the boundary line between the two currents being a line of minimum or maximum height of sea-

[1] A. COLDING: Nogle Undersögelser over Stormen over Nord- og Mellem-Europa af 12:te—14:de Nov. 1872 og over den derved fremkaldte Vandflod i Østersöen. Danske Vidensk. Selskabs Skrifter, Natur. og Math. Afd. Vol. I. N:o 4, 1881.

From manuscript charts kept at the Meteorological Institute at Christiania it may be seen that the Storm began sometime between 8 a. m. Nov. 11 and 8 a. m. Nov. 12.

level. As a matter of fact, if a region of uniform wind stretches right across the sea, the surface current produced would in the absence of a surface-inclination, steadily carry water transversely across the wind-belt, from one part of the ocean to the other. If this flow be not compensated by another surface-current, there must arise an inclination of surface perpendicularly to the length-direction of the wind-region, and a midwater-current in the direction of the latter. Thus for instance we may expect within the Trade Wind's regions and the regions of the westwinds in the north and south Pacific, a midwater-current running approximately in the direction of the parallel of latitude. The equatorial counter-current is naturally explained as a compensation-current produced by the accumulation of water by the equatorial midwater-currents.

On top of these deep midwater-currents running nearly constantly in directions determined by the average wind-conditions of the season or of the year, the surface-current down to the depth D, fluctuates with the varying winds. The fact that the wind-drifts actually fluctuate in all directions and can be recognized only in their *mean* movement, is thus explained, and especially as the variations of the wind velocity are often greater than that of the average wind itself.

IV. Calculation of the quantity *D*.

The magnitude of D or »*Depth of frictional influence*» is the key which must be found, before the theory here given can be made fully applicable. There are two methods which would seem convenient for the determination of D.

1:) By direct observation of the direction of the wind-current at different depths. This may be carried out by an application of Equation (11) p. 17 far away from land and just after the rise of a new wind, before any storing up of the water has had time to occur. Or the observations may be made in an enclosed sea after the stationary state of wind-current has become established; to which Problem *b* p. 37 may be applied.

2:) By observation of the velocity of the surface-current produced by a given wind. In the calculation it is then necessary to give the most careful attention possible to the direction and velocity of the midwater-current.

With the observations which are at present at our disposal, the second method is the only possible one. The calculation which is given here below is made chiefly as an example, and with the object of giving some indication at any rate of the *order of magnitude* of the quantity concerned.

Equations (5) and the results obtained in Problems *a* and *b* above, give the relation between the velocity of the surface-current and the tangential pressure T of the wind. We must then, first of all find the relation between T and the wind velocity; and this can be done by help of observations on the storing up of water (»Windstau»). If there were no earth's rotation at all it is easily seen that in the case of stationary motion, and if the total flow in any direction is *nil*, the inclination γ of the water-surface would be given by

$$\sin\,\gamma = \frac{3}{2}\frac{T}{qgd}\,, \tag{28}$$

T, q, g, d, having the same significations as before. As was mentioned on p. 37 the final inclination of the surface will in the case of an enclosed sea, be only slightly influenced by the earth's rotation, particularly when d/D is small; and we may therefore use Equation (28) in this case and if necessary, afterwards apply a correction.

From the observations on the coasts of the Baltic during the Storm of November 12th—14th 1872, COLDING has found (or verified) a relation[1] which with our notations may be written

$$h = 14450\,\sqrt{d\,\sin\,\gamma}\,, \tag{29}$$

h being the wind velocity in cm. per second and d the depth of the water in cm. For our present purpose we may use this formula without further scrutiny. It gives together with (28) if we assume $qg = 1000$, the relation

$$T = 0{,}0000032\;h^2 \tag{30}$$

between the wind-velocity in cm. per second and its tangential pressure T in gram . cm^{-1} . sec^{-2}.

---

[1] A. COLDING l. c. p. 38.

The relationship between h and the velocity of the drift-current is much more uncertain. The most reliable observations are undoubtedly those made during the drift of the »Fram» in the years 1893—96[1]. From these it was found that the velocity of the ice-drift in cm. per second was approximately 1,9 times the wind-velocity in m. per second; but as this result refers to a sea covered with ice, it would not *a priori* be fair to apply it to the case of the open sea.

On the other hand the »Charts of meteorological Data for Nine Ten-Degree Squares 20° N—10° S etc.» edited by the Meteorological Office in London, are not very suitable for our purpose on account of the low latitude of the region considered. MOHN[2] selected from these tables those cases in which wind and current had approximately the same direction; this selection, though it might have been justifiable in the light of the theory of that time, is from the present point of view, obviously quite arbitrary and may introduce considerable errors. The result of Mohn's calculation is — when in accordance with KRÜMMEL's recommendation KÖPPEN's table instead of SCOTT's table is used for converting Beaufort-scale into m. per second — that the velocity of the drift current in cm. per second should be 4,7 times the wind velocity in m. per second. It is worth while remarking that this value and that from the »Fram's» drift are not at all in disagreement with one another; according to the theory the numbers 1,9 and 4,7 should be in the inverse ratio of the square roots of the sines of the corresponding latitudes, and this is actually true if we assume the mean latitudes to be 82° and 9°,3. This is for several obvious reasons only a *possibility* of agreement and not necessarily a direct agreement. We may however for the present use the above-mentioned result and assume the relation $V = 0{,}019/\sqrt{\sin\varphi}\,.\,h$ between the surface-current velocity V and the wind velocity h, both in cm. per second.

A closer examination of the influence of coasts and neighbouring currents in the cases on which NANSEN's and MOHN's calculations are based, would carry us far beyond the limits of the present communication; we may content ourselves with a mean value of this influence. It was mentioned on

[1] FRIDTJOF NANSEN l. c. p. 2.
[2] H. MOHN l. c. p. 1.

p. 33 that the surface velocity of a stationary wind current outside an infinitely long coast would as an average be 1,66 times the value V_0 (p. 7) which it would have in the absence of coasts. Further, it was remarked (p. 38) that this value actually must be somewhat reduced, since the length of the coast cannot be regarded as infinite.

We may then as a fairly probable value assume $V = 1{,}5\ V_0$ which gives

$$V_0 \sqrt{\sin \varphi} = 0{,}0127\, h, \tag{31}$$

and as a result of (30)

$$T = 0{,}00025\ h V_0 \sqrt{\sin \varphi}\,.$$

From (5) and (6) it follows

$$V_0 = \frac{\pi T}{\sqrt{2 D q \omega \sin \varphi}},$$

and if the above value be substituted for T in this equation, it gives

$$D = \frac{0{,}00025\, \pi h}{q \omega \sqrt{2 \sin \varphi}} = \frac{7{,}6}{\sqrt{\sin \varphi}} h,$$

where D and h may obviously be measured in meters and in meters per second respectively[1]. *I. e. at high latitudes the Depth of Wind-current should be as many meters as the wind velocity in m. per second, multiplied by 7 or 8; at lower latitudes it should be greater in the ratio* $1 : \sqrt{\sin \varphi}$. With a trade-wind of 7 m. per second it would at 15° latitude be about 100 m. At 45° latitude it would with the same wind be 60 or 70 m; with a gale of 17 m. per second 150 m. and with a gentle breeze of 4,5 m. per second 40 m.

It will scarcely be necessary to emphasise the fact that these numbers are merely intended to give an idea of the

---

[1] Tho average wind-velocity during the Storm on the Baltic studied by COLDING, was 20 or 25 m. per second, which makes $D = 200$ m. about. The mean depths of the sections examined were never more than 100 m. and as a rule below 70 m. The correction for the influence of the earth's rotation, on the storing up of the water is then according to p. 37 quite insignificant.

From the expression for D it would follow that the coefficient of friction μ in a wind-current is about 4 times the square of the wind-velocity in m. per second. Thus for instance it would during a trade wind of 7 m. per sec, be about 200 C. G. S.

order of magnitude of the quantity under consideration, and that they are not to be regarded as results of exact calculation.

Note on the Law of Friction in the Sea.

In the preceding analysis μ has always been regarded as a *constant* (though with the reservation that different values may have to be given to it under different circumstances). This wants justification if the theory should be followed up to *numerical calculation*. It is obvious that μ cannot generally be regarded as a constant when the density of the water is not uniform within the region considered. For μ will be greater within the layers of uniform density and comparatively small within the transition-layers where the formation of vortices must be much reduced owing to the differences of density. This is however a complication which cannot be taken into account in the general theory; this is more suitably left to the applications.

On the other hand it seems probable, that the irregular vortex-motion, and therefore also μ, would increase in proportion with the rate of gliding of one water-layer above the other, and would consequently, in the case of wind-currents, gradually diminish from the surface downwards. It seems likely that this circumstance might considerably modify the results and in particular introduce errors in the calculation of the quantity D. A comparison is therefore made below, between the results obtained above with the »*linear* friction-relationship» and those which would be obtained from the »*quadratic* relationship», *i. e.* if the frictional forces were proportional to the *square* of the rate of gliding[1].

To solve the problem of wind-currents with the *quadratic* friction-relationship, put

---

[1] The latter friction-relationship can certainly not hold *exactly* for a real viscous fluid. For it implies that the virtual value of μ vanishes with the velocity-derivatives $\partial u/\partial x$, $\partial v/\partial x$ etc.; while actually the value of μ cannot fall below the real coefficient of viscosity, as measured in capillary tubes for instance. The latter quantity may however on account of its smallness be altogether left out of consideration in this connection.

$$\mu = \nu \sqrt{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2},$$

where ν is now to be regarded as a constant. The components of tangential stress are then

$$(32)\quad T_x = -\nu \sqrt{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2}\,\frac{du}{dz};\quad T_y = -\nu \sqrt{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2}\,\frac{dv}{dz},$$

and we have instead of Equations (3) p. 6,

$$(33)\quad \begin{cases} \dfrac{\frac{du}{dz}}{\sqrt{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2}}\left(\dfrac{du}{dz}\dfrac{d^2u}{dz^2} + \dfrac{dv}{dz}\dfrac{d^2v}{dz^2}\right) + \sqrt{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2}\,\dfrac{d^2u}{dz^2} + 2a^2v = 0 \\[2ex] \dfrac{\frac{dv}{dz}}{\sqrt{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2}}\left(\dfrac{du}{dz}\dfrac{d^2u}{dz^2} + \dfrac{dv}{dz}\dfrac{d^2v}{dz^2}\right) + \sqrt{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2}\,\dfrac{d^2v}{dz^2} - 2a^2u = 0 \end{cases}$$

where

$$a = \sqrt{\frac{q\omega \sin \varphi}{\nu}}.$$

The treatment of these equations is (in the case of d infinite) largely simplified by the obvious fact, that the angle α between the directions of velocity and of tangential stress must in this case be a constant, so that $d\alpha/dz = 0$. If the angles β and γ be so chosen (Fig. 9) that

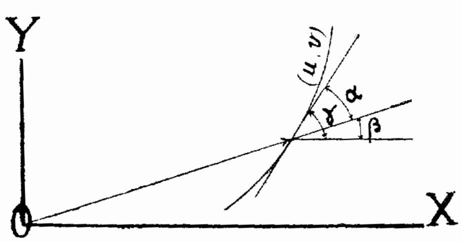


Fig. 9.

$$\sin \beta = \frac{v}{\sqrt{u^2+v^2}};\quad \cos \beta = \frac{u}{\sqrt{u^2+v^2}}$$

$$\sin \gamma = \frac{-\frac{dv}{dz}}{\sqrt{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2}};\quad \cos \gamma = \frac{-\frac{du}{dz}}{\sqrt{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2}},$$

so is $\alpha = \gamma - \beta$, and as α is a constant,

$$\frac{d\beta}{dz} = \frac{d\gamma}{dz}.$$

For a value of z which makes $v=0$, $\beta=0$ and consequently $\alpha=\gamma$, this equation becomes

$$\frac{1}{u}\frac{dv}{dz} = \frac{\frac{du}{dz}\frac{d^2v}{dz^2} - \frac{dv}{dz}\frac{d^2u}{dz^2}}{\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2},$$

or

$$(34) \qquad \frac{1}{u}\left(\frac{dv}{dz}\right)^3 = \left(\frac{du}{dz}\frac{d^2v}{dz^2} - \frac{dv}{dz}\frac{d^2u}{dz^2}\right)\sin^2\alpha .$$

For the same value of z, Equations (33) may be written in the form

$$\frac{du}{dz}\left[(1+\cos^2\alpha)\frac{d^2u}{dz^2} + \cos\alpha \sin\alpha \frac{d^2v}{dz^2}\right] = 0$$

$$\frac{du}{dz}\left[\cos\alpha \sin\alpha \frac{d^2u}{dz^2} + (1+\sin^2\alpha)\frac{d^2v}{dz^2}\right] + 2a^2u\cos\alpha = 0 .$$

When solved for $\frac{du}{dz}\frac{d^2u}{dz^2}$ and $\frac{du}{dz}\frac{d^2v}{dz^2}$ and after multiplying the former quantity by $\frac{dv}{dz}\Big/\frac{du}{dz} = \tan\alpha$, they give

$$\frac{dv}{dz}\frac{d^2u}{dz^2} = a^2u\sin^2\alpha\cos\alpha$$

$$\frac{du}{dz}\frac{d^2v}{dz^2} = -a^2u\cos\alpha\,(1+\cos^2\alpha)$$

and consequently

$$\frac{du}{dz}\frac{d^2v}{dz^2} - \frac{dv}{dz}\frac{d^2u}{dz^2} = -2a^2u\cos\alpha .$$

If this value be substituted on the right hand side of (34), we find

$$\left(\frac{dv}{dz}\right)^3 = -2a^2u^2\cos\alpha\sin^2\alpha = -2a^2u^2\operatorname{cotan}\alpha\sin^3\alpha .$$

On dividing by $\tan^3\alpha$ and by u^3 respectively, changing u for V and taking the cube-root, this equation gives the two equations

$$\frac{dV}{dz} = -\cos\alpha \sqrt[3]{2\,a^2 \operatorname{cotan}\alpha}\; V^{2/3}$$

$$\frac{d\beta}{dz} = -\sin\alpha \sqrt[3]{2\,a^2 \operatorname{cotan}\alpha}\; V^{-1/3},$$

which obviously hold for any value of z, if V denotes the absolute velocity $\sqrt{u^2+v^2}$ at the depth z. By integration we find

$$V = V_0\,(1-Bz)^3 \tag{35}$$
$$\beta = 3\tan\alpha \log(1-Bz) + C,$$

where

$$B = \frac{1}{3}\cos\alpha \sqrt[3]{\frac{2\,a^2 \operatorname{cotan}\alpha}{V_0}},$$

V_0 = the velocity of the surface-water, and C is a constant. The values

$$u = V\cos\beta;\quad v = V\sin\beta$$

satisfy Equations (33) identically, if

$$\cos\alpha = \sqrt{\frac{3}{7}};\ \sin\alpha = \sqrt{\frac{4}{7}},\quad i.\ e.\ \alpha = 49^\circ{,}1,$$

and the contention $da/dz=0$ is then justified.

As is seen from the first equation (35) the velocity of the wind-current as well as its two first derivatives would in the case of the quadratic friction-relationship, become exactly *nil* even at a finite depth $z = 1/B$, and it is obviously *nil* from that depth right down to the bottom. It might then seem natural to denote the depth $1/B$ as the depth of wind-current. It is however more suitable for a faithful comparison of the two theories to denote by the depth of wind-current the quantity

$$D' = 0{,}8/B.$$

With this notation and when the axis of positive y is laid in the wind's direction, we finally have

$$u = V_0\left(1 - \frac{0{,}8}{D'}z\right)^3 \cos\beta$$

$$v = V_0\left(1 - \frac{0{,}8}{D'}z\right)^3 \sin\beta \tag{36}$$

$$\beta = 3\tan\alpha \log\left(1 - \frac{0{,}8}{D'}z\right) + \frac{\pi}{2} - \alpha.$$

$$\alpha = 49^\circ{,}1.$$

Equations (36) contain the exact solution of our problem as long as the depth d is not smaller than $1{,}25\ D'$. It is represented in Fig. 10 (the full-drawn line) for $z=0$, $z=0{,}1\ D'$, $z=0{,}2\ D'$ etc. in exactly the same way as Equations are (5) in Fig. 1, p. 8. To facilitate the comparison the solution (5) is represented in the same Fig. 10 with dotted lines, V_0 and winddirection being the same in the case of both curves. With the degree of accuracy which would come into question in the oceanography of the present, the difference between the two curves may certainly be regarded as insignificant. It implies in the main a slightly greater angle of deflection of the current in the case of the quadratic friction-relationship than in the case of the linear one.

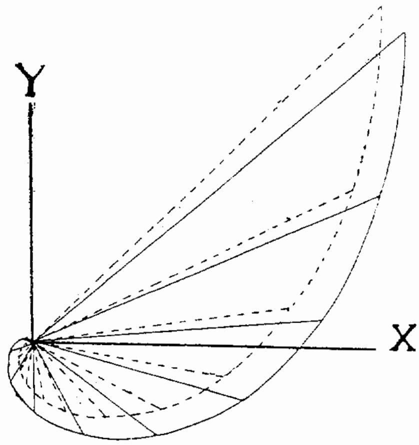


Fig. 10.

The relationships between the depth and velocity of the wind-current on the one hand and the tangential pressure T of the wind on the other, are somewhat different according to the different laws of friction. Equations (36) give

$$\left(\frac{dv}{dz}\right)_{z=0} = -\frac{0{,}8\sqrt{21}\,V_0}{D'}$$

and consequently

(37)
$$T = \nu\left(\frac{dv}{dz}\right)^2_{z=0} = \frac{13{,}44\ \nu V_0^2}{D'^2}.$$

According to definition

(38)
$$D' = \frac{0{,}8}{B} = 3{,}05\sqrt[3]{\frac{\nu V_0}{q\omega \sin\varphi}}.$$

By elimination first of V_0 and then of D' between (37) and (38) we get

(39)
$$D' = 2{,}79\,\frac{\sqrt[4]{\nu T}}{\sqrt{q\omega \sin\varphi}}$$

(40)
$$V_0 = 0{,}76\,\frac{T^{3/4}}{\sqrt[4]{\nu}\sqrt{q\omega \sin\varphi}}$$

It is clear that the influence of the earth's rotation upon this result would be of the same order of smallness as in the case of the linear friction-relationship, if d/D is small. From (41) and the empirical formula (29) which is based upon direct observation, we find in the same way in which equation (30) was deduced:

$$T = 0{,}00000328\, h^2.$$

Equation (31) between V_0 and h would not hold in the case of the quadratic relationship, since as mentioned above the latter would make V_0 proportional to $T^{3/4}$ or to $h^{3/2}$. To follow up the comparison as far as possible we may however restrict ourselves to the case of an average wind-velocity for which the empirical formula (31) still holds, and then

$$T = 0{,}000258\, h V_0 \sqrt{\sin\varphi}.$$

From (37) and (38)

$$V_0 = \frac{2{,}12\, T}{D' q \omega \sin\varphi}$$

and finally after elimination of T by help of the preceding equation,

$$D' = \frac{0{,}000547\, h}{q\omega\sqrt{\sin\varphi}} = \frac{h}{7{,}5\sqrt{\sin\varphi}},$$

i. e. almost exactly the same result as with the linear friction-relationship.

The only important real difference between the results of the two friction-relationships would, according to the above calculations, be in the connection between the depth and velocity of the wind-current on the one hand and the wind-velocity on the other hand. In other respects the theory of wind-currents in deep water is even formally independent of the choice made; neither does this choice essentially affect the calculation of the quantity D corresponding to a given wind-velocity. In the case of convection-currents, the question considered will certainly be of inferior importance owing to the much greater anomalies connected with the irregular distribution of density. On the other hand the choice of friction-relationship might possibly be of somewhat greater influence in the case of wind-currents in shallow water, though it does not seem probable. The quadratic relationship would however in this case make the theory much too complicated and difficult.

Although from a logical point of view it would be more consequent to adopt the quadratic friction-relationship than to use the linear one with a variable coefficient of friction, the latter alternative being the simpler is therefore for the present to be preferred.

C. A. Bjerknes' Experiment.

The late Prof. C. A. BJERKNES at Kristiania, whose vivid interest seems to have been bestowed on every extension of knowledge in his branch, also made in the autumn of 1902, some experiments with the object of verifying some of the results to be found in Section I of this paper.

His apparatus consisted of a low cylinder (12 or 17 cm. high and 36 or 44 cm. wide) made of metal or glass, and resting on a table which could be put into uniform rotation (about 7 turns a minute against the sun) by means of a water-turbine. To the upper edge of the rotating cylinder was attached a jet having a horizontal split, 10 cm. long and 1 mm. broad; and through this a stream of air was forced from a pump to produce a wind diametrically across the cylinder over a 10 cm. broad belt.

The motion of the water was observed by means of small balls which just floated in it. As a result of the rotation of the cylinder the current always swept towards the right and thus formed a large whirl-pool occupying, when seen in the wind's direction, the middle as well as the right half of the cylinder.

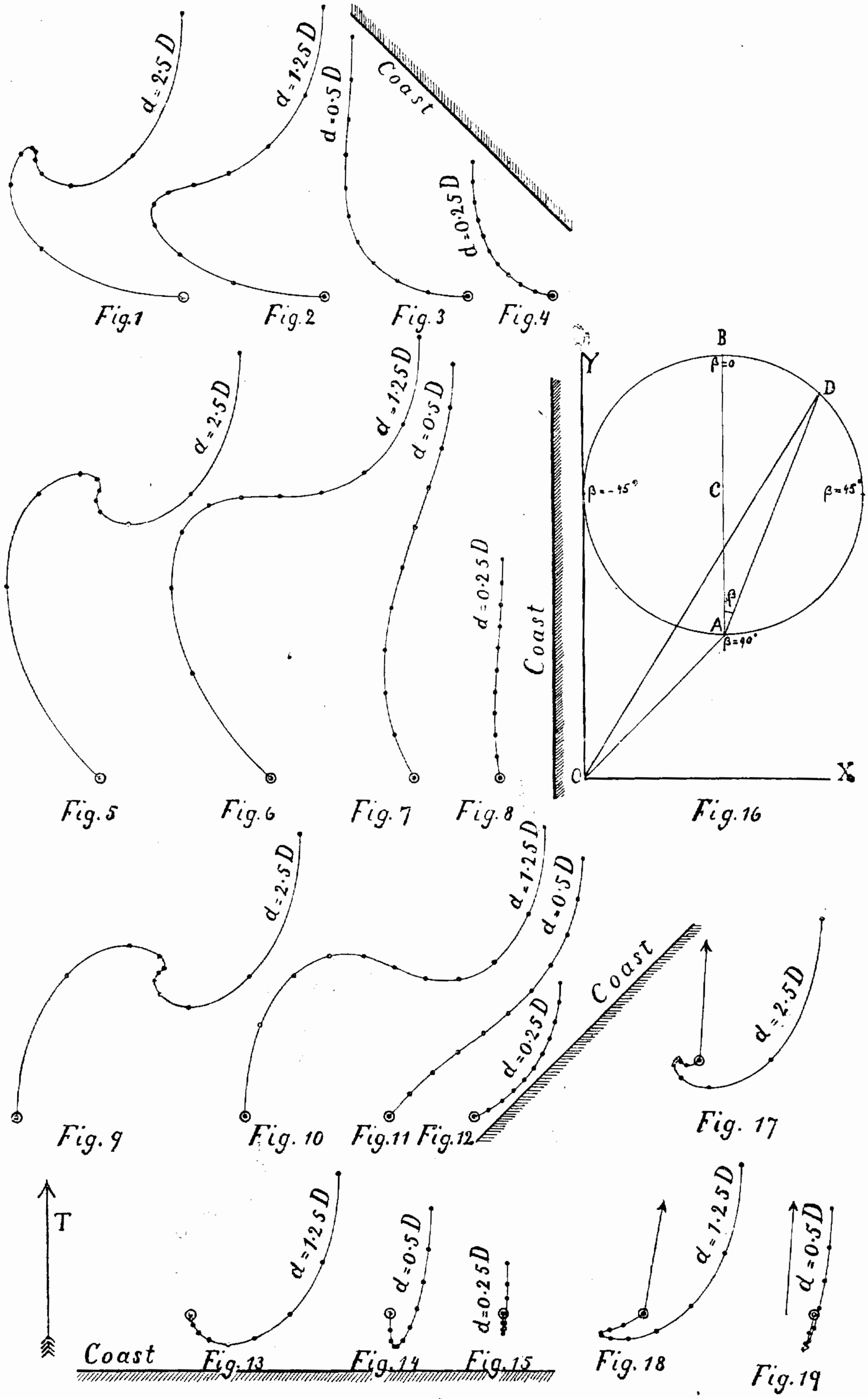
The direction of motion at different depths was observed at the centre of the cylinder, on a sensitive vane (4 cm. long and 1 mm. high) which could be raised and lowered during the experiment without disturbing the motion. The direction of the vane was read against a glass square divided into radians and laid on top of the cylinder. The following table taken from Prof. BJERKNES' note-book kept in his laboratory gives as an example a series of measurements made during such an experiment. The first column gives the depth in cm. below the surface, the second column the deviation of the current to the right or left of the wind's direction.

Surface	20—25°	right
0,5 cm.	45—50°	»
1 »	45—50°	»
1,5 »	25—30°	»
2 »	0—10°	»
2,5 »	0°	»
3 »	5—10°	left
3,5 »	5—10°	»
4 »	5—10°	»
4,5 »	10°	»
5 »	10—15°	»
5,5 »	15—20°	»
6 »	20°	»
6,5 »	20°	»
7 »	20—25°	»

The circumstances under which these experiments were made were in any case such as to satisfy the conditions for stationary motion but very roughly; and an exact interpretation may furthermore be difficult owing to the shape of the vessel etc. It is certain however that their real object was to obtain a merely qualitative verification of the reality of the phenomena considered, and as such they are very striking and instructive. Both the deflection of the surface-current and the increase of the angle of deflection downwards, are quite apparent. The angle of deflection increases only in the very uppermost layer; and this is explained as a result of the rapid rotation of the vessel. Indeed a value of $\mu = 0,3$ (which would not appear to be too small for motion on such a small scale) would give $D = 2$ cm. only. The directions of motion below this level have very much the appearance of a »midwater-current» produced by a pressure-gradient.

Tryckt den 14 september 1905.

Uppsala 1905. Almqvist & Wiksells Boktryckeri-A.-B.


d = 2·5 D
d = 1·25 D
d = 0·5 D
Coast
d = 0·25 D
Fig. 1
Fig. 2
Fig. 3
Fig. 4
d = 2·5 D
d = 1·25 D
d = 0·5 D
d = 0·25 D
Coast
Y
B
β = 0
D
β = −45°
C
β = 45°
β
A
β = 90°
O
X
Fig. 5
Fig. 6
Fig. 7
Fig. 8
Fig. 16
d = 2·5 D
d = 1·25 D
d = 0·5 D
d = 0·25 D
Coast
d = 2·5 D
Fig. 9
Fig. 10
Fig. 11
Fig. 12
Fig. 17
T
d = 1·25 D
d = 0·5 D
d = 0·25 D
d = 1·25 D
d = 0·5 D
Coast
Fig. 13
Fig. 14
Fig. 15
Fig. 18
Fig. 19